Poznámka k analýze stability v neautonomních systémech

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Outline

1. Introduction
2. Example
3. A more general picture
4. Conclusion

Outline

RD on growing domains

Self-organisation models: Turing (mathematical); Allen-Cahn, Cahn-Hilliard (physical, free energy)
The dimensional model is given by

$$\partial_t u + \nabla_x.(\mathbf{a}u) = D_u \nabla_x^2 u + f(u,v)$$
$$\partial_t v + \nabla_x.(\mathbf{a}v) = D_v \nabla_x^2 v + g(u,v),$$

with $x \in \Omega := [0, L(t)]$ and **a** is the velocity vector induced by domain growth.
Zero-flux boundary conditions at the boundary $\partial\Omega(t)$.

RD on growing domains. Lagrangian frame

Non-dimensional form $(U_0,\ t = L_0^2/D_u\tau,\ L_0 = L(0))$

$$\partial_\tau \mathbf{u} + h(\tau)\mathbf{u} = \frac{1}{\varphi^2(\tau)}\mathbf{D}\Delta_\xi \mathbf{u} + \gamma \mathbf{F}_*(\mathbf{u}).$$

where $h(\tau) = [1/L]\partial L/\partial\tau$ is the non-dimensional expansion rate of the domain and

$$\varphi(\tau) = \exp\int_0^\tau h(q)\mathrm{d}q = \frac{L(t(\tau))}{L_0},$$

and

$$\mathbf{D} = \mathrm{diag}(1, d), d = D_v/D_u \geq 1 \quad \gamma = \omega L_0^2/D_u.$$

Outline

RD on growing domains. Expansion

Exponential growth $\varphi(\tau) = e^{rt}$, $h(\tau) = r$ and with $D = dD_u = D_v/\mu^2$ we have

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} + r \begin{pmatrix} u \\ v \end{pmatrix} = e^{-2rt} D \begin{pmatrix} \mu^2 & 0 \\ 0 & 1 \end{pmatrix} \partial_{xx} \begin{pmatrix} u \\ v \end{pmatrix} + \mathbf{J} \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{for} \quad x \in (0,1),$$

$$\partial_x \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ at } x = 0, 1,$$

We focus on the amplitude equations for each mode $\cos(kx)$, $k = n\pi$

$$\partial_t \begin{pmatrix} p \\ q \end{pmatrix} = e^{-2rt} \mathbf{M} \begin{pmatrix} p \\ q \end{pmatrix} + \mathbf{J}_r \begin{pmatrix} p \\ q \end{pmatrix} \qquad \text{with} \qquad \mathbf{M} = \begin{pmatrix} -\mu^2 k^2 D & 0 \\ 0 & -k^2 D \end{pmatrix}$$

and where $\mathbf{J}_r = \mathbf{J} - r\mathbf{I}$.

RD on growing domains. Expansion II

Let $F(t) = \int_0^t \mathrm{d}s(\varphi(s))^{-2} = \frac{1}{2r}\left(1 - e^{-2rt}\right)$ and

$$\begin{pmatrix} a \\ b \end{pmatrix} = \exp\left[-F(t)\mathbf{M}\right]\begin{pmatrix} p \\ q \end{pmatrix}.$$

As

$$\begin{pmatrix} a \\ b \end{pmatrix} \geq \begin{pmatrix} p \\ q \end{pmatrix} \geq \begin{pmatrix} \exp(-\mu^2 k^2 D\frac{1}{2r}) & 0 \\ 0 & \exp(-k^2 D\frac{1}{2r}) \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix}$$

we have that $(p, q)^T$ decays if and only if $(a, b)^T$ decays.

Amplitude problem. Non-autonomous ODE I

The evolution equation for (a, b) is

$$
\begin{aligned}
\partial_t \begin{pmatrix} a \\ b \end{pmatrix} &= \exp\left[-F(t)\mathbf{M}\right].\mathbf{J}_r.\exp\left[F(t)\mathbf{M}\right].\begin{pmatrix} a \\ b \end{pmatrix} \\
&= \begin{pmatrix} J_{11} - r & J_{12}\exp[(\mu^2-1)\kappa^2(t))] \\ J_{21}\exp[(1-\mu^2)\kappa^2(t))] & J_{22} - r \end{pmatrix}.\begin{pmatrix} a \\ b \end{pmatrix},
\end{aligned}
$$

where $\kappa^2(t) = k^2 DF(t)$. Equivalently

$$
\begin{aligned}
\partial_t^2 a = {} & (J_{11} - r)\partial_t a \\
& + J_{12}J_{21}\left[(\mu^2-1)k^2 D(\varphi(t))^{-2} + (J_{22} - r)\right](\partial_t a - (J_{11} - r)a),
\end{aligned}
$$

with b given in terms of a and its time derivative, using the first equation.

Amplitude problem. Non-autonomous ODE II

Instead of special functions we reduce the order (linear ODE)

$$a(t) = a_0(t)\left(C_1 + \int_0^t \mathrm{d}s v(s)\right), \quad v(t) = C_2 \frac{1}{a_0(t)^2} \exp\left(-\int_0^t \mathrm{d}s (\varphi(s))^{-2}\right)$$

where $a_0(t)$ is a particular solution of the equation and C_1, C_2 are integration constants.
We choose $C_1 = 1/a_0(0)$ so that perturbations initiate from 1.

Amplitude problem. Particular example I

Consider $r = 1/2,\ \mu = 1/10,\ D = 100/99$ and

$$\mathbf{J} = \begin{pmatrix} 3/2 & -2 \\ 3 & -7/2 \end{pmatrix},$$

Then a particular solution is

$$a_0(t) = \exp[-3t + e^{-t}k^2]k^2 + 2\exp[-2t + e^{-t}k^2]$$

and the general solution reads

$$a(t) = a_0(t)\left(\frac{e^{-k^2}}{k^2+2} + C_2\int_0^t \mathrm{d}s \frac{\exp(-k^2 e^{-s} + s)}{k^2(k^2 e^{-s} + 2)^2}\right).$$

It is instructive to expand the solution about $t = 0$

$$a(t) \approx 1 + t\frac{C_2 - k^6 - 5k^4 - 4k^2}{(k^2+2)k^2} + O(t^2)$$

to see that first few modes of the perturbations grow initially even though they later decay.

Amplitude problem. Particular example II

Now consider growth rate $r = 1/6$. A particular solution is

$$a_0(t) = \exp\left[3k^2 e^{-\frac{1}{3}t}\right]\Big(81k^{10}e^{-\frac{10}{3}t} + 1080k^8 e^{-3t} + 5040k^6 e^{-\frac{8}{3}t} + 10080k^4 e^{-\frac{7}{3}t} + 8400k^2 e^{-2t} + 2240e^{-\frac{5}{3}t}\Big),$$

and the general solution for $t \sim O(1)$ and $k \gg 1$ is

$$\begin{aligned} a(t) &\approx 81C_2k^{10}e^{-\frac{10}{3}t}\exp\left[3k^2e^{-\frac{1}{3}t}\right]\int_0^t \mathrm{d}s \exp\left[-3k^2e^{-\frac{1}{3}s}\right]e^{\frac{13}{3}s} \\ &= 81C_2k^{10}e^{-\frac{10}{3}t}e^{\frac{1}{\delta}g(t)}\int_0^t \mathrm{d}s\ e^{-\frac{1}{\delta}g(s)}e^{\frac{13}{3}s} \end{aligned}$$

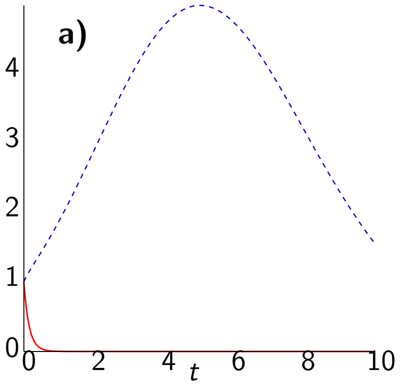
with $\delta = 1/[3k^2] \ll 1$ and $g(t) = e^{-t/3}$. Finally, using Laplace's method for $t \gg \delta$ we have

$$a(t) \approx 81C_2k^{10}e^t e^{tg'(t)/\delta}\int_0^t \mathrm{d}s\ e^{-sg'(t)/\delta} \approx 81C_2k^8e^{4t/3}.$$

sensitivity to initial noise; non-linear system can pick up transient growth; transient growth can be more extensive as k increases (breakdown of the model)

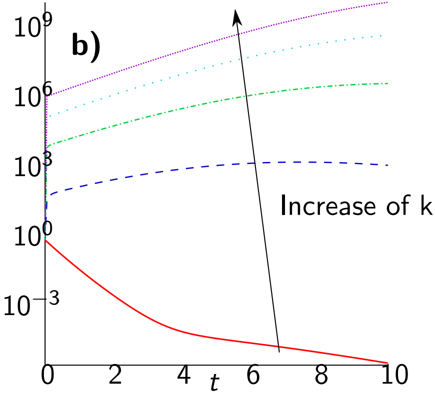
Figures I

$C_2 = 0$ (red solid) and $C_2 = 10^{-3}$ with $k = 2$ (blue dashed)



Figures II

$C_2 = 10^{-3}$ for $k = 1$ (red solid), $k = 3$ (dark blue dashed), $k = 5$ (green dash-dotted), $k = 7$ (light blue dotted) and $k = 9$ (purple densely dotted)



Outline

General uniform growth. Nonautonomous ODE

For a general (smooth enough) uniform growth, dimension, (smooth bounded) domain we have an amplitude equation of the form We establish some growth bounds on general second order non-autonomous ODE of the form

$$\ddot{Y} + F(t)\dot{Y} + G(t)Y = 0\,. \tag{1}$$

We shall show that

Thrm. Let $\Phi \in C^2(\mathbb{R})$ such that $\Phi(t) > 0$ for all $t \in \mathcal{I}$. Consider the ODE (1) *and suppose that*

$$G(t) \leq -\frac{\ddot{\Phi}}{\Phi} - \frac{\dot{\Phi}}{\Phi}F(t), \quad t \in \mathcal{I}. \tag{2}$$

Then, (1) *has a fundamental solution $Y(t)$ with $|Y(t)| \geq \Phi(t)$ for all $t \in \mathcal{I}$.*

Proof of the lower bound I

Proof. *Equality:* Φ *is a solution.*
Next, consider $G(t) = -\frac{\ddot\Phi}{\Phi} - \frac{\dot\Phi}{\Phi}F(t) - H(t)$ *for some* $H(t) \geq 0$*.*
Then, (1) *reads*

$$\ddot Y + F(t)\dot Y - \left(\frac{\ddot\Phi}{\Phi(t)} + \frac{\dot\Phi}{\Phi(t)}F(t) + H(t)\right) Y = 0. \tag{3}$$

The change of variable $Y(t) = Y(t_0)\exp\left(\int_{t_0}^t Z(s)ds\right)$ *(with choice* $Y(t_0) = \Phi(t_0)$ *and* $\dot Y(t_0) = \dot\Phi(t_0)$*) we transform* (3) *into*

$$\dot Z = -Z^2 - F(t)Z + \frac{\ddot\Phi}{\Phi} + \frac{\dot\Phi}{\Phi}F(t) + H(t) \geq Z^2 - F(t)Z + \frac{\ddot\Phi}{\Phi} + \frac{\dot\Phi}{\Phi}F(t) =: \dot Z_1. \tag{4}$$

Proof of the lower bound II

It is easy to verify that $Z_1(t) = \dot{\Phi}(t)/\Phi(t)$.
By differential inequality (4) *and since* $Z(t_0) = Z_1(t_0)$, *we have* $Z(t) \geq Z_1(t)$ *for all* $t \in \mathcal{I}$ *and hence*

$$Y(t) = Y(t_0)\exp\left(\int_{t_0}^{t} Z(s)ds\right) \geq Y(t_0)\exp\left(\int_{t_0}^{t} Z_1(s)ds\right) = \Phi(t).$$

Then, for this choice of initial data, $|Y(t)| \geq \Phi(t)$ *for all* $t \in \mathcal{I}$.

Corollary

Consider $\Phi(t) = \exp(\delta t)$ for some $\delta > 0$. From theorem, we have

$$G(t) \le -\frac{\ddot{\varphi}}{\varphi} - 2\delta\frac{\dot{\varphi}}{\varphi} - \delta^2 - \left(\frac{\dot{\varphi}}{\varphi} + \delta\right) F(t).$$

Taking $\delta \to 0^+$, and strict inequality, we recover the weakest bound for exponential growth during $t \in \mathcal{I}$,

$$G(t) < -\frac{\ddot{\varphi}}{\varphi} - \frac{\dot{\varphi}}{\varphi} F(t), \quad \text{for all} \quad t \in \mathcal{I}.$$

Weaker (polynomial) lower bounds are available as well.

Implications for TI on growing domains

- Sufficient condition for TI on growing domains which exactly reduces to classical TI conditions on static domains
- $D_u/D_v \neq 1$ for TI
- transient behaviour analysis and its dependence on k can be analysed using the criterion:
  when
  $$J_{j,j}(d_j - d_{\neg j}) \leq \frac{\dot{\mu}}{\mu}(d_j - d_{\neg j}).$$
  large enough modes become unstable. For fast enough growth this inequality is satisfied and hence fast growth always yields transient exponential growth for large wavenumbers.
- no need for slow growth; history dependence

Outline

Summary

- Non-autonomous (ODEs) have some distinct qualitative features and are hard to analyse
- Transient growth seems to be an easier problem than large time behaviour
- application to TI

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