Metamaterial transitions on curved manifolds

Bachelor thesis

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Metamaterials

Negative Index Metamaterial: $\epsilon, \mu < 0$

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Negative permitivity ϵ - plasma physics, first NIM with both $\epsilon, \mu < 0$ designed in 1999 by Pendry **et al**[1].

[1] John B Pendry, Anthony J Holden, David J Robbins, and WJ Stewart. Magnetism from conductors and enhanced nonlinear phenomena. IEEE transactions on microwave theory and techniques, 47(11):2075–2084, 1999.

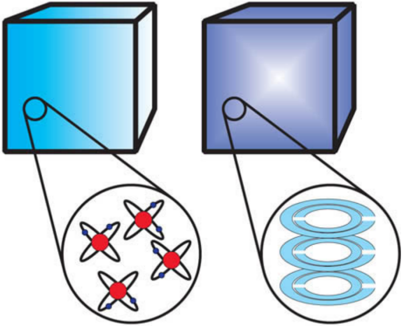


Figure: The conventional material (left) derives its properties from atoms. However, in the metamaterial (right) the role of the atoms is now played by small sub-units.[2]

[2] John B Pendry. Negative refraction. Contemporary Physics, 45(3):191–202, 2004.

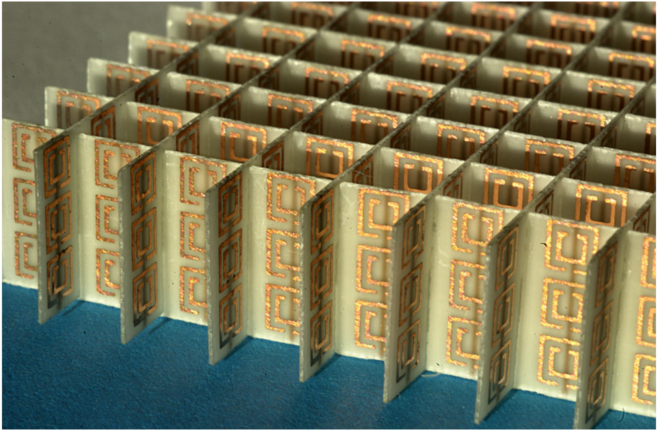


Figure: A split ring structure on copper circuit board with copper wires to give both $\epsilon, \mu < 0$[3].

[3] John B Pendry. Negative refraction. Contemporary Physics, 45(3):191–202, 2004.

V. Veselago

The first one to considered both $\epsilon, \mu < 0$ was Russian physicist Victor Veselago in 1967[4]. The refractive index stays positive and real, so he asked if there would be any difference at all.

$$v = \frac{c}{n}, \qquad n = \sqrt{\epsilon\mu}.$$

[4]Viktor G Veselago. Electrodynamics of substances with simultaneously negative and. Usp. Fiz. Nauk, 92:517, 1967.

Effects

Poynting vector $\vec{S} = \frac{1}{4\pi}\vec{E} \times \vec{H}$ usually points in direction of $\vec{k}$, but in metamaterial it points opposite to $\vec{k}$. This leads to interesting effects such as reversed Cherenkov radiation cone, superlensing and metamaterial *cloaking*.

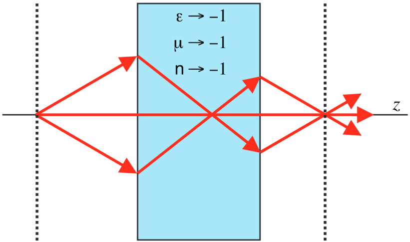


Figure: Medium bends light to a negative angle relative to the surface normal. Released from the medium, the light reaches a focus for a second time in the image plane.[5]

[5] John B Pendry. Negative refraction. Contemporary Physics, 45(3):191–202, 2004.

Mathematical model

Quasi-static approximation will be considered. Next, only electric field will be investigated.

$$\operatorname{div} \vec{D} = \rho, \qquad \operatorname{rot} \vec{E} = 0$$

and $\vec{E}$ is represented by potential $\vec{E} = -\operatorname{grad} \varphi$.

Further substituting $\vec{D} = \epsilon \vec{E}$:

$$-\operatorname{div}(\epsilon \operatorname{grad} \varphi) = \rho.$$

Reminder

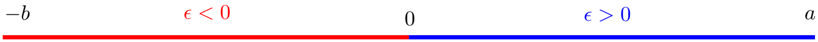
Sobolev spaces

$$H^k(\Omega) = \left\{ f \in L^2(\Omega) \mid \forall |\alpha| < k,\ \exists f^{(\alpha)} \text{ weakly}, \|f^{(\alpha)}\|_{L^2} < \infty \right\}$$

A composite string

Consider a string composed of both metamaterial and a regular material[6]

$$\epsilon(x) = \begin{cases} \epsilon_+, & \text{if } x \in (0, a), \\ -\epsilon_-, & \text{if } x \in (-b, 0). \end{cases}$$



[6] Amru Hussein. Sign-indefinite second-order differential operators on finite metric graphs.

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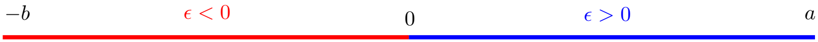

$$-\operatorname{div} \epsilon \operatorname{grad} \varphi \xrightarrow{\text{1D}} -(\epsilon \varphi')'$$

To overcome problems with ill-posedness because of ϵ discontinuity at interface, we resctrict possible φ with condition:

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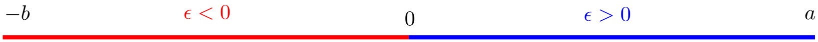

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To overcome problems with ill-posedness because of ϵ discontinuity at interface, we resctrict possible φ with condition:

$$\begin{aligned} \varphi_+ &:= \varphi \restriction (0, a) & \varphi_- &:= \varphi \restriction (-b, 0) \\ \varphi_+(0^+) &= \varphi_-(0^-) & \epsilon_+\varphi_+'(0^+) &= -\epsilon_-\varphi_-'(0^-), \end{aligned}$$

and set $\varphi_+ \in H^2((0, a))$ and $\varphi_- \in H^2((-b, 0))$.

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Impose Dirichlet boundary conditions and set

$$\Omega_+ = (0, a), \quad \Omega_- = (-b, 0).$$

Then we can define operator A_1 on $L^2((-b, a))$:

[7] Filip Hložek. Operator theoretic approach to the theory of metamaterials. BSc. thesis, 2014.

[8] Sabina, Zairova. Časový vývoj metamateriálových strun. BSc. thesis. České vysoké učení technické v Praze. Vypočetní a informační centrum., 2019.

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$$\operatorname{dom} A_1 = \left\{ \varphi = \begin{pmatrix} \varphi_+ \\ \varphi_- \end{pmatrix} \in H^2(\Omega_+) \oplus H^2(\Omega_-) \;\middle|\; \begin{array}{l} \varphi_-(-b) = 0 = \varphi_+(a), \\ \varphi_+(0) = \varphi_-(0), \\ \epsilon_+ \varphi_+'(0) = -\epsilon_- \varphi_-'(0) \end{array} \right\}$$

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This operator is **selfadjoint**[7][8] and $\sigma_{\mathrm{ess}}(A_1) = \varnothing$.

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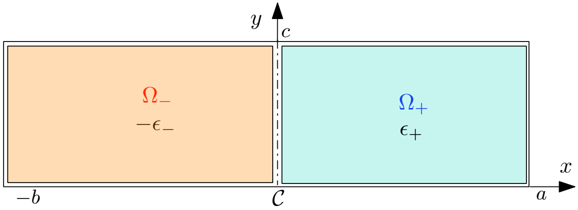
A composite rectangle



$$\Omega_+ = (0, a) \times (0, c), \quad \Omega_- = (-b, 0) \times (0, c), \quad \mathcal{C} = \{0\} \times (0, c),$$
$$\Omega = \Omega_+ \cup \mathcal{C} \cup \Omega_- \subset \mathbb{R}^2$$

[9] Jussi Behrndt and David Krejčiřík. An indefinite Laplacian on a rectangle. Journal d'Analyse Mathématique, 134(2):501–522, 2018.

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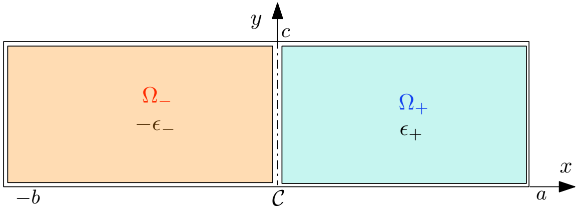

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$-\operatorname{div}(\epsilon \operatorname{grad} f)$ for constant $\epsilon_+, \epsilon_- > 0$ with Dirichlet boundary condition.

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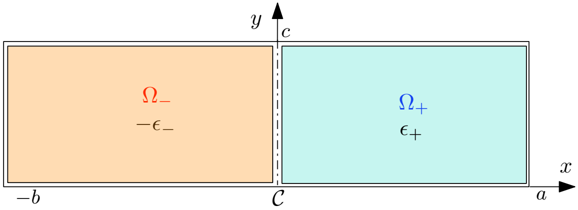

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$-\operatorname{div}(\epsilon \operatorname{grad} f)$ for constant $\epsilon_+, \epsilon_- > 0$ with Dirichlet boundary condition.

This case was examined in 2014[9] for $\epsilon_- = \epsilon_+$.

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A composite rectangle

It turns out that operator A_2 on $L^2(\Omega)$

$$A_2 \begin{pmatrix} f_+ \\ f_- \end{pmatrix} = \begin{pmatrix} -\epsilon_+ \Delta f_+ \\ \epsilon_- \Delta f_- \end{pmatrix},$$

$$\operatorname{dom} A_2 = \left\{ f = \begin{pmatrix} f_+ \\ f_- \end{pmatrix} \in H^2(\Omega_+) \oplus H^2(\Omega_-) \;\middle|\; \begin{array}{l} f_\pm|_{\partial\Omega} = 0, \\ f_+|_{\mathcal{C}} = f_-|_{\mathcal{C}}, \\ \epsilon_+ \partial_{\mathbf{n}} f_+|_{\mathcal{C}} = \epsilon_- \partial_{\mathbf{n}} f_-|_{\mathcal{C}} \end{array} \right\},$$

is **essentially selfadjoint** and $0 \in \sigma_{\mathrm{ess}}(A_2) \iff \epsilon_+ = \epsilon_-$.

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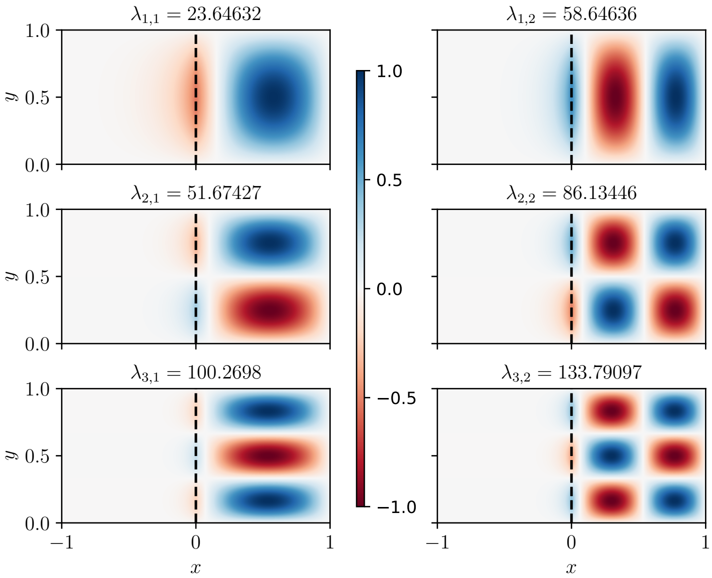
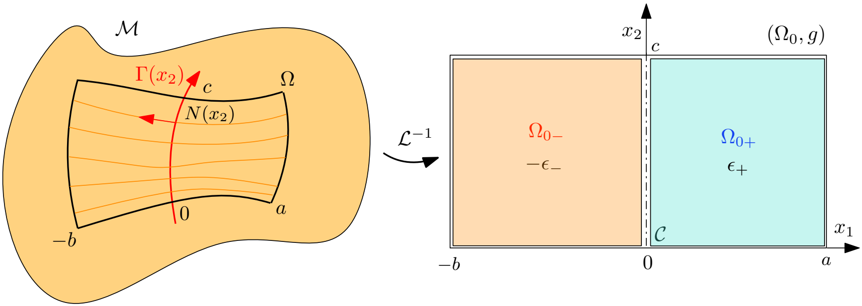
0 is infinitely degenerate eigenvalue $\iff$ $a = b$, otherwise it is an accumulation point of $\sigma(A_2)$.



$\lambda_{1,1} = 23.64632$
$\lambda_{1,2} = 58.64636$
$\lambda_{2,1} = 51.67427$
$\lambda_{2,2} = 86.13446$
$\lambda_{3,1} = 100.2698$
$\lambda_{3,2} = 133.79097$
y
x
1.0
0.5
0.0
−0.5
−1.0
−1
0
1

Tubular neigbourhoods

2D Riemannian manifold $\mathcal{M}$, geodesic curve $\Gamma : (0, c) \to \mathcal{M}$, $|\dot{\Gamma}(x_2)| = 1$



$\Omega := \mathcal{L}(\Omega_0)$, $\mathcal{L}(x_1, x_2) := \exp_{\Gamma(x_2)}(x_1 N(x_2))$.[10]

[10] David Krejčiřík and Petr Siegl. $\mathcal{PT}$-symmetric models in curved manifolds. Journal of Physics A: Mathematical and Theoretical, 43(48):485204, 2010.

induced metric on (Ω_0, g): $(g_{ij})=\begin{pmatrix} 1 & 0 \\ 0 & f^2 \end{pmatrix}$, Jacobi equation for f:

$$\partial_1^2 f + Kf = 0 \quad \wedge \quad \begin{cases} f(0,\cdot) = 1, \text{ on interface } \mathcal{C} \\ \partial_1 f(0,\cdot) = -\kappa = 0 \end{cases}$$

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and for constant K

$$f(x_1,x_2) = \begin{cases} \cos(\sqrt{K}x_1) & \text{if } K > 0, \\ 1 & \text{if } K = 0, \\ \cosh(\sqrt{|K|}x_1) & \text{if } K < 0. \end{cases}$$

Operator B_K

Laplace-Beltrami on (Ω_0, g) :

$$-\operatorname{div}(\epsilon \operatorname{grad} \varphi) = -\frac{1}{f}\partial_1 \left(\epsilon f \partial_1 \varphi\right) - \frac{1}{f}\partial_2 \left(\epsilon \frac{1}{f} \partial_2 \varphi\right).$$

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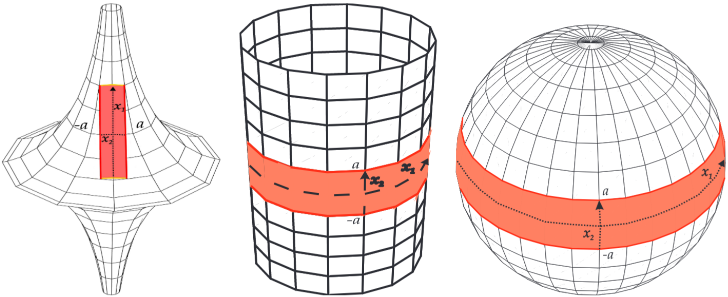
Define B_K on $L^2(\Omega_0, f\,\mathrm{d}x)$

$$B_K \begin{pmatrix} \varphi_+ \\ \varphi_- \end{pmatrix} = \begin{pmatrix} -\epsilon_+ \Delta_g \varphi_+ \\ \epsilon_- \Delta_g \varphi_- \end{pmatrix}$$

and domain will be very similar to 2D case.

$$B_K \overset{1-1}{\longleftrightarrow} |K| B_{\text{sgn}(K)}$$



(a) $K = -1$, pseudosphere (b) $K = 0$, cylinder (c) $K = 1$, sphere

Figure: [11]

[11] $\mathcal{PT}$-symmetric models in curved manifolds. Journal of Physics A: Mathematical and Theoretical, 43(48):485204, 2010.

Separation of variables

$$\varphi(x_1, x_2) = \sum_{m=1}^{\infty} \psi_m(x_1)\phi_m(x_2)$$

$$B_K\varphi = \sum_{m=1}^{\infty} (B_K^m \psi_m) \otimes \phi_m,$$

$$\phi_m(x_2) = \sqrt{\frac{2}{c}} \sin(\frac{m\pi}{c} x_2)$$

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$$B_K^m := \begin{pmatrix} \epsilon_+ \\ -\epsilon_- \end{pmatrix} \cdot \begin{cases} -\partial_1^2 + \tan(x_1)\partial_1 + \frac{(m\pi)^2}{c^2 \cos^2(x_1)}, & \text{if } K = 1, \\ -\partial_1^2 + (\frac{m\pi}{c})^2, & \text{if } K = 0, \\ -\partial_1^2 - \tanh(x_1)\partial_1 + \frac{(m\pi)^2}{c^2 \cosh^2(x_1)}, & \text{if } K = -1. \end{cases}$$

Solutions

Solutions for $B_1^m \psi_\pm = \lambda \psi_\pm$

$$\begin{cases} \pm\epsilon_\pm(-\psi_\pm'' + \tan(x)\psi_\pm' + \frac{(m\pi)^2}{c^2\cos^2(x)}\psi_\pm) = \lambda\psi_\pm, \\ \psi_+(a) = 0 = \psi_-(-b), \\ \psi_+(0+) = \psi_-(0-), \\ \epsilon_+\psi_+'(0+) = -\epsilon_-\psi_-'(0-). \end{cases}$$

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Solution given in terms of associated Legendre $P_\nu^{(\mu)}, Q_\nu^{(\mu)}$ functions.

If we set $\lambda = 0$, $a = b$, solution is given in terms elementary functions.

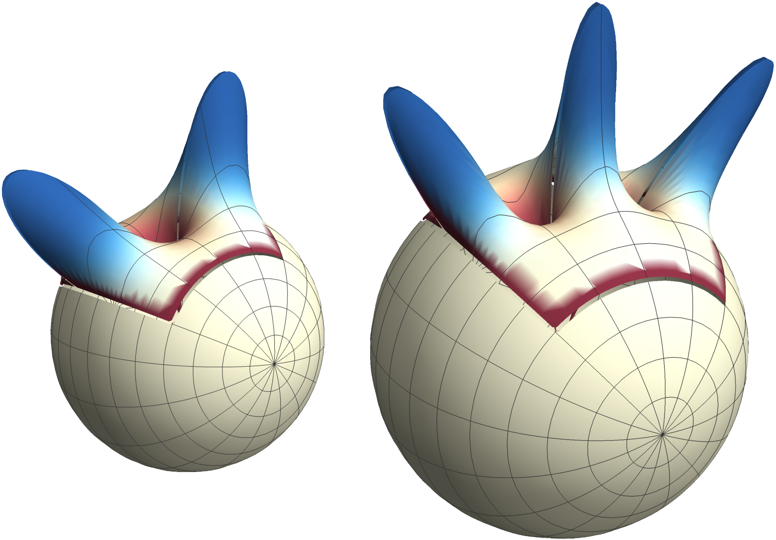

Figure: $K = 1$

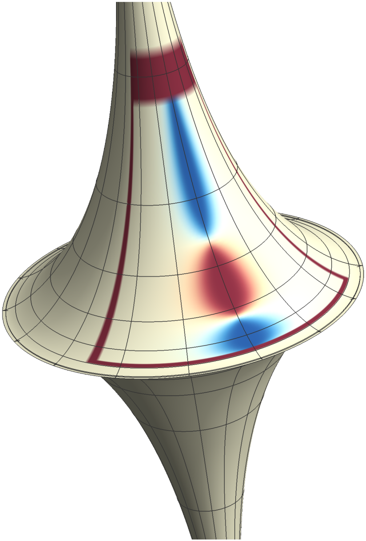

Figure: $K = -1$

Unitary transformation for $K = 1$

$$U_{+1} : L^2\left((-b,a), \mathrm{d}x_1\right) \to L^2\left((-b,a), \cos(x_1)\,\mathrm{d}x_1\right),$$
$$(U_{+1}\psi)(x) := \cos(x)^{-\frac{1}{2}}\psi(x)$$
$$U_{+1}^{-1} B_1^m U_{+1} = A_{\mathrm{1D}} + V_{+1}$$

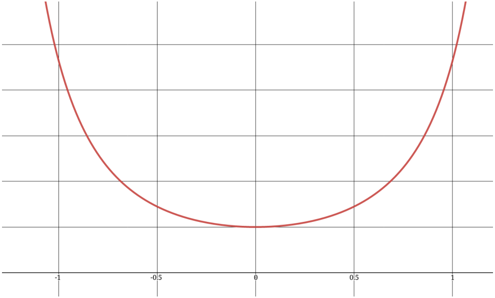

Figure: Multiplicative potential V_{+1}.

Properties

$$B_1^m \overset{U_{+1}}{\longleftrightarrow} A_{1\mathrm{D}} + V_{+1}$$

$\Longrightarrow \mathrm{B}_1^m$ selfadjoint $\Longrightarrow \mathrm{B}_1$ **essentially selfadjoint**,

$$B_{-1}^m \overset{U_{-1}}{\longleftrightarrow} A_{1\mathrm{D}} + V_{-1}$$

$\Longrightarrow \mathrm{B}^m{}_{-1}$ selfadjoint $\Longrightarrow \mathrm{B}_{-1}$ **essentially selfadjoint**,

$\Longrightarrow \mathrm{B}_K$ **essentially selfadjoint**.

Properties

In 2D: A_{2D} essentially selfadjoint
Now: ✓

In 2D: $0 \in \sigma_{\mathrm{ess}}(A_{\mathrm{2D}}) \iff \epsilon_+ = \epsilon_+$.
Now: ? (probably)

In 2D: $\lambda = 0$ is infinitely degenerate eigenvalue $\iff a = b \wedge \epsilon_+ = \epsilon_-$.
Now: ✓

Proof of A_1 self-adjointness

A_1 is symmetric: $(\phi, A_1\psi) = (A_1\phi, \psi)$ for $\phi, \psi \in \operatorname{dom} A_1$.

Now for per-partes ($\phi \in \operatorname{dom} A_1^*, \psi \in \operatorname{dom} A_1$) we need $\phi', \phi'' \in \operatorname{dom} A_1^*$. For this we define restriction $\dot{A}_1 \subset A_1$:

$$\operatorname{dom} \dot{A}_1 = \left\{ \varphi = \begin{pmatrix} \varphi_+ \\ \varphi_- \end{pmatrix} \in H^2(\Omega_+) \oplus H^2(\Omega_-) \;\middle|\; \begin{array}{l} \varphi_-(-b) = 0 = \varphi_+(a), \\ \varphi_+(0) = 0 = \varphi_-(0), \\ \epsilon_+\varphi_+'(0) = 0 = -\epsilon_-\varphi_-'(0) \end{array} \right\}$$

then[12]

$$\operatorname{dom} \dot{A}_1^* = \left\{ \varphi = \begin{pmatrix} \varphi_+ \\ \varphi_- \end{pmatrix} \in H^2(\Omega_+) \oplus H^2(\Omega_-) \mid \varphi_-(-b) = 0 = \varphi_+(a) \right\}.$$

[12] Kato, Tosio. Perturbation theory for linear operators. Vol. 132. Springer Science Business Media, 2013.

Theorem

Let A be a symmetric operator on Hilbert space $\mathcal{H}$ and $\{\psi_n\}_{n=0}^{\infty}$ orthonormal basis in $\mathcal{H}$. Then if for each $n \in \mathbb{N}$ holds: $\psi_n \in \operatorname{dom} A$ and there exists λ_n s.t. $A\psi_n = \lambda_n \psi_n$, then A is essentially self-adjoint. Spectrum of $\overline{A}$ is closure $\overline{\sigma(A)}^{\mathbb{R}}$ in $\mathbb{R}$.

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