Spectrum of the Ekman boundary layer problem

Based on joint work with O. Ibrogimov and P. Siegl

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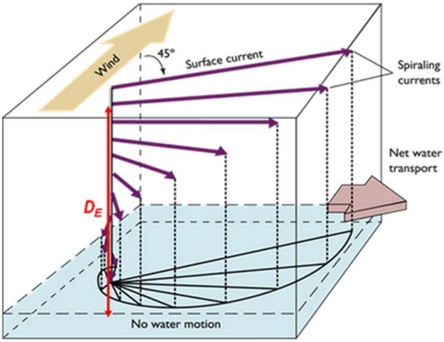
Outline

- introduction to physical setting
- mathematical formulation of corresponding spectral problem
- previous research
- solution of open problem in Greenberg and Marletta, 2004
- illustration on 1D Schrödinger operator
- challenges in Ekman problem

Ekman boundary layer problem - fluid dynamics



(a) EKMAN SPIRAL IN THE NORTHERN HEMISPHERE

`www.offshoreengineering.com`

- 1893-96 Nansen (NO)
- 1902 Ekman (SE)
- model phenomena with Navier-Stokes equation

- "Surface currents, the Ekman spiral, and Ekman transport" (Youtube, SciencePrimer)

Ekman boundary layer problem - model

Navier-Stokes equations for incompressible viscous flow in rotating frame (see e.g. Hess, Hieber, Mahalov, and Saal, 2010)

$$\left.\begin{aligned} \partial_t u - \nu\Delta u + \Omega e_3 \times u + (u\cdot\nabla)u + \nabla p = 0 \\ \operatorname{div} u = 0 \end{aligned}\right\} \quad \nu > 0,\ \Omega \in \mathbb{R}$$

for velocity vector u and pressure p with boundary conditions

$$\begin{aligned} u(t, x_1, x_2, 0) &= (0,0,0), & t > 0,\ \ x_1, x_2 \in \mathbb{R} \\ u(t, x_1, x_2, x_3) &\to (u_\infty, 0, 0), & x_3 \to \infty,\ \ u_\infty \geq 0 \end{aligned}$$

Ekman spiral (stationary solution) with $\delta = (2\nu/\Omega)^{\frac{1}{2}}$

$$u_E(x_3) = u_\infty(1 - \mathrm{e}^{-x_3/\delta}\cos(x_3/\delta), \mathrm{e}^{-x_3/\delta}\sin(x_3/\delta), 0)$$

$$p_E(x_2) = -\Omega u_\infty x_2$$

Ekman boundary layer problem - linearisation

linearise around Ekman spiral → linear Cauchy problem

associated spectral problem transforms to

$$\begin{cases} ((-\partial^2+\alpha^2)^2 + \mathrm{i}\alpha RU(-\partial^2+\alpha^2) + \mathrm{i}\alpha RV'')\phi + 2\partial\psi = \lambda(-\partial^2+\alpha^2)\phi \\ 2\partial\phi + (\mathrm{i}\alpha RU' + (-\partial^2+\alpha^2) + \mathrm{i}\alpha RV)\psi = \lambda\psi \end{cases}$$

system of ODEs on $\mathbb{R}_+$ with boundary conditions

$$\phi(0) = \phi'(0) = \psi(0) = 0$$
$$\phi(\infty) = \phi'(\infty) = \psi(\infty) = 0$$

- after transformation $u_E \to (U, V, 0)$
- Reynolds number $R = u_\infty \delta/\nu \geq 0$
- wave number $\alpha > 0$

Reformulation of spectral problem

$$\mathcal{A} = \begin{pmatrix} (-\partial^2+\alpha^2)^2 + \mathrm{i}\alpha R V(-\partial^2+\alpha^2) + \mathrm{i}\alpha R V'' & 2\partial \\ 2\partial + \mathrm{i}\alpha R U' & -\partial^2+\alpha^2+\mathrm{i}\alpha R V \end{pmatrix}$$

$$\mathcal{B} = \begin{pmatrix} -\partial^2+\alpha^2 & 0 \\ 0 & I \end{pmatrix}$$

family of non-self-adjoint operator matrices in $\mathcal{H} = L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+)$

$$\begin{cases} \mathcal{T}(\lambda) = \mathcal{A} - \lambda\mathcal{B}, \qquad \lambda \in \mathbb{C} \\ \mathrm{Dom}\mathcal{T} = \{(f_1, f_2) \in H^4(\mathbb{R}_+) \times H^2(\mathbb{R}_+) \,:\, f_1(0) = f_1'(0) = f_2(0) = 0\} \end{cases}$$

with more general coefficients

$$V, V', V'', U' \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$$

question: structure and location of the spectrum

$$\sigma(\mathcal{T}) = \{\lambda \in \mathbb{C} \,:\, 0 \in \sigma(\mathcal{T}(\lambda))\}, \qquad \mathrm{Re}\,\sigma(\mathcal{T}) \geq 0\ ?$$

Some previous research

- experiments with rotating tank
  (Faller, 1963)
- non-rigorous stability and spectral analysis
  (Lilly, 1966; Spooner, 1982)
  - calculated essential spectrum non-rigorously
  - numerical computations, location of eigenvalues for Ekman spiral and different α and R
  - non-rigorous domain truncation
- stability analysis using PDE techniques
  (Giga et al., 2007; Giga and Saal, 2015; Hess, Hieber, Mahalov, and Saal, 2010)

Some previous research

- essential spectrum using singular sequences, spectral enclosure (Greenberg and Marletta, 2004)

$$\sigma_{\text{ess}}(\mathcal{T}) = \{\lambda \in \mathbb{C} \,:\, \exists\, \xi \in \mathbb{R},\ p_\lambda(\xi) = 0\}$$

$$p_\lambda(\xi) = (\xi^2 + \alpha^2)(\xi^2 + \alpha^2 - \lambda)^2 + 4\xi^2$$

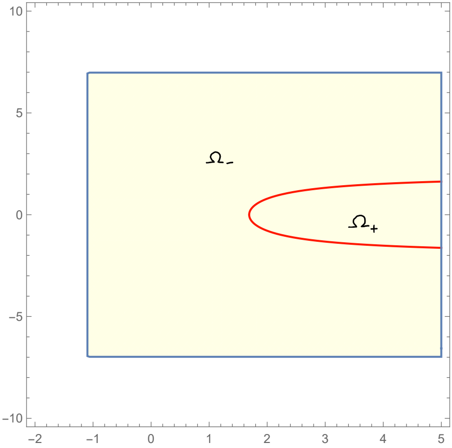
$$\sigma(\mathcal{T}) \subset \{\lambda \in \mathbb{C} \,:\, \operatorname{Re}\lambda \geq \gamma,\ |\operatorname{Im}\lambda| \leq \eta\} = S$$

  where γ, η depend on α, R, U, V (L^∞-norms)

- essential spectrum with abstract operator theoretic approach (Marletta and Tretter, 2007)

Greenberg and Marletta, 2004

red curve $\sigma_{\text{ess}}(\mathcal{T})$, yellow half-strip S



- $\mathbb{C} \setminus \sigma_{\text{ess}}(\mathcal{T}) = \Omega_+ \cup \Omega_-$
- eigenvalues discrete in Ω_-
- what happens in Ω_+ ?
- domain truncation spectrally exact if no open sets of eigenvalues exist
- justifies non-rigorous approach in Spooner, 1982 and Lilly, 1966

red curve $\sigma_{\rm ess}(\mathcal{T})$, yellow half-strip S

Greenberg and Marletta, 2004

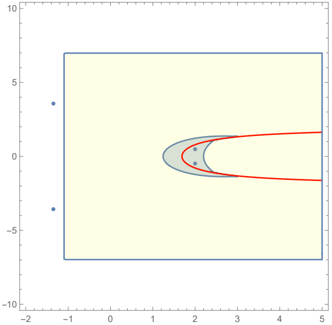
Theorem (G.-Ibrogimov-Siegl, 2020)

Let $\Omega = \mathbb{C} \setminus \sigma_{\rm ess}(\mathcal{T})$, *then* $\sigma_{\rm p}(\mathcal{T}) \cap \Omega$ *is discrete and bounded. Moreover,*

$$\sigma_{\rm p}(\mathcal{T}) \cap \Omega \subset \{\lambda \in \Omega : \alpha R\, \mathrm{r}(\mathcal{Q}(\lambda)) \geq 1\} \qquad (1)$$

where $\mathcal{Q}(\lambda)$, $\lambda \in \Omega$ *is a certain family of HS integral operators and*

$$\mathrm{r}(\mathcal{Q}(\lambda)) = \mathcal{O}(|\lambda|^{-\frac{1}{2}}), \quad \lambda \to \infty \text{ in } \Omega.$$



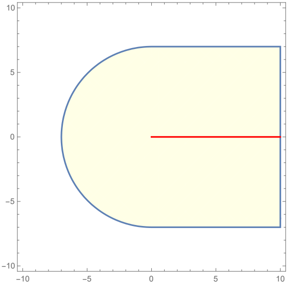
blue enclosure (1)

(estimate in terms of α *and* L^1*-norms of coefficients)*

1D Schrödinger operator on $\mathbb{R}_+$ with Dirichlet BC

Hilbert space $L^2(\mathbb{R}_+)$, potential $V \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$

$$\begin{cases} T = \underbrace{-\partial^2}_{=L} + V \end{cases} \qquad \mathrm{Dom}\, T = \mathrm{Dom}\, L = \{f \in H^2(\mathbb{R}_+) \,:\, f(0) = 0\}$$



red curve $\sigma_{\mathrm{ess}}(T)$, yellow enclosure (2)

relatively compact perturbation

$$\sigma_{\mathrm{ess}}(L) = \sigma_{\mathrm{ess}}(T) = [0, \infty)$$

$$\sigma(T) \setminus [0, \infty) \subset \sigma_{\mathrm{p}}(T)$$

estimating numerical range

$$\sigma(T) \subset \{\lambda \in \mathbb{C} \,:\, \mathrm{dist}(\lambda, [0, \infty)) \leq \|V\|_{L^\infty(\mathbb{R}_+)}\} \tag{2}$$

1D Schrödinger: Birman-Schwinger principle

pioneering work by Abramov, Aslanyan, and Davies, 2001, developed extensively within non-self-adjoint spectral theory

- $V = V_2 V_1$ with $V_1 = |V|^{\frac{1}{2}} \in L^2(\mathbb{R}_+)$, $V_2 = V/V_1 \in L^2(\mathbb{R}_+)$
- Birman-Schwinger operator

$$Q(\lambda) = V_1 (L - \lambda)^{-1} V_2, \quad \lambda \in \rho(L) = \mathbb{C} \setminus [0, \infty)$$

- then $-1 \in \rho(Q(\lambda)) \iff \lambda \in \rho(T)$ and

$$(T - \lambda)^{-1} = (L - \lambda)^{-1} (I - V_2 [I + Q(\lambda)]^{-1} V_1 (L - \lambda)^{-1})$$

- need Green's function of unperturbed operator
- investigate when $I + Q(\lambda)$ is invertible (e.g. $\|Q(\lambda)\| < 1$)

1D Schrödinger: Birman-Schwinger principle

resolvent of free Schrödinger for $\lambda \in \rho(L) = \mathbb{C} \setminus [0, \infty)$

$$(L - \lambda)^{-1} g = \int_{\mathbb{R}_+} \mathcal{L}_\lambda(\cdot, y) g(y) \mathrm{d}y, \qquad g \in L^2(\mathbb{R}_+)$$

with $k^2 = \lambda, \ \operatorname{Im} k < 0$ the integral kernel reads

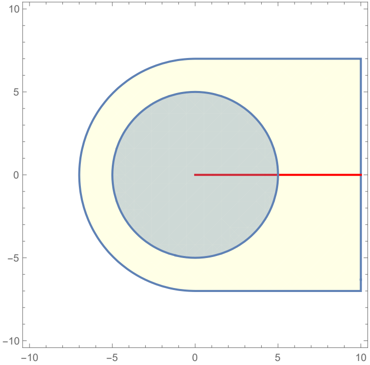
$$\mathcal{L}_\lambda(x, y) = \mathcal{G}_\lambda(x - y) - \mathcal{G}_\lambda(x + y) = \frac{1}{2\mathrm{i}k} \left(\mathrm{e}^{-\mathrm{i}k|x-y|} - \mathrm{e}^{-\mathrm{i}k|x+y|} \right)$$

recall BS-operator $Q(\lambda) = V_1 (L - \lambda)^{-1} V_2$, $\lambda \in \rho(L)$

$$\begin{aligned} \|Q(\lambda)\| &\leq \|V_1(x) \mathcal{L}_\lambda(x, y) V_2(y)\|_{L^2(\mathbb{R}_+^2)} \\ &\leq \|\mathcal{L}_\lambda\|_{L^\infty(\mathbb{R}_+^2)} \|V_1\|_{L^2(\mathbb{R}_+)} \|V_2\|_{L^2(\mathbb{R}_+)} \leq \frac{1}{|\lambda|^{\frac{1}{2}}} \|V\|_{L^1(\mathbb{R}_+)} \end{aligned}$$

enclosure for the point spectrum

$$\sigma_{\mathrm{p}}(T) \setminus [0,\infty) \subset \{\lambda \in \mathbb{C} : |\lambda| \le \|V\|^2_{L^1(\mathbb{R}_+)}\} \qquad (3)$$



red curve $\sigma_{\mathrm{ess}}(T)$, yellow enclosure (2), blue enclosure (3)

Ekman spectral problem - some challenges

$$\mathcal{H} = L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+), \qquad V, V', V'', U' \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$$

$$\mathcal{T}(\lambda) = \overbrace{\begin{pmatrix} (-\partial^2+\alpha^2)(-\partial+\alpha^2-\lambda) & 2\partial \\ 2\partial & -\partial^2+\alpha^2-\lambda \end{pmatrix}}^{=\mathcal{L}(\lambda)} + \mathrm{i}\alpha R \underbrace{\begin{pmatrix} V(-\partial^2+\alpha^2)+V'' & 0 \\ U' & V \end{pmatrix}}_{=\mathcal{V}}$$

$$\mathrm{Dom}\mathcal{L} = \mathrm{Dom}\mathcal{V} = \mathrm{Dom}\mathcal{T} = \{(f_1, f_2) \in H^4(\mathbb{R}_+) \times H^2(\mathbb{R}_+) \; : \; f_1(0) = f_1'(0) = f_2(0) = 0\}$$

- spectral problem for operator matrix family
- more boundary conditions to consider
- perturbation is differential operator

Ekman spectral problem - some challenges

$$\mathcal{L}(\lambda)^{-1}G = \int_{\mathbb{R}_+} \mathcal{L}_\lambda(\cdot, y)G(y)\,\mathrm{d}y, \quad G \in \mathcal{H},\ \lambda \in \Omega$$

where (with $x, y \in \mathbb{R}_+$)

$$\mathcal{L}_\lambda(x,y) = \begin{pmatrix} \mathcal{G}_{11}(x-y) + \mathcal{G}_{11}(x+y) & \mathcal{G}_{12}(x-y) - \mathcal{G}_{12}(x+y) \\ \mathcal{G}_{12}(x-y) + \mathcal{G}_{12}(x+y) & \mathcal{G}_{22}(x-y) - \mathcal{G}_{22}(x+y) \end{pmatrix}$$

$$+ \frac{2}{\mathcal{G}_{11}(0)} \begin{pmatrix} -\mathcal{G}_{11}(x)\mathcal{G}_{11}(y) & \mathcal{G}_{11}(x)\mathcal{G}_{12}(y) \\ -\mathcal{G}_{12}(x)\mathcal{G}_{11}(y) & \mathcal{G}_{12}(x)\mathcal{G}_{12}(y) \end{pmatrix}$$

and $\mathcal{G}_\lambda = (\mathcal{G}_{ij}) = \mathcal{F}^{-1}[\mathcal{M}_\lambda^{-1}]$ with

$$\mathcal{M}_\lambda^{-1}(\xi) = \frac{1}{p_\lambda(\xi)} \begin{pmatrix} \xi^2 + \alpha^2 - \lambda & 2\mathrm{i}\xi \\ 2\mathrm{i}\xi & (\xi^2+\alpha^2)(\xi^2+\alpha^2-\lambda) \end{pmatrix}$$

$$p_\lambda(\xi) = (\xi^2+\alpha^2)(\xi^2+\alpha^2-\lambda)^2 + 4\xi^2$$

Ekman spectral problem - some challenges

since $\lambda \in \Omega$ there are three roots of p_λ with

$$\operatorname{Im}\mu_j < 0, \quad j = 1, 2, 3$$

if no roots of p_λ are multiple (i.e. $\lambda \in \Omega \setminus B_\alpha$) then with

$$c_j = \frac{1}{(\mu_j^2 - \mu_{(j+1)_{\mathrm{mod}\,3}}^2)(\mu_j^2 - \mu_{(j+2)_{\mathrm{mod}\,3}}^2)}$$

the Green's function reads

$$\mathcal{G}_{11}(x) = -\frac{\mathrm{i}}{2}\sum_{j=1}^{3}\frac{c_j}{\mu_j}(\mu_j^2 + \alpha^2 - \lambda)\, e^{-\mathrm{i}\mu_j|x|}$$

$$\mathcal{G}_{12}(x) = \mathcal{G}_{21}(x) = \operatorname{sgn} x \sum_{j=1}^{3} c_j\, e^{-\mathrm{i}\mu_j|x|}$$

$$\mathcal{G}_{22}(x) = -\frac{\mathrm{i}}{2}\sum_{j=1}^{3}\frac{c_j}{\mu_j}(\mu_j^2 + \alpha^2 - \lambda)(\mu_j^2 + \alpha^2)\, e^{-\mathrm{i}\mu_j|x|}$$

Ekman spectral problem - some challenges

$$\mathcal{V}_2 = \begin{pmatrix} W_1 & 0 \\ 0 & W_1 \end{pmatrix}, \quad \mathcal{V}_1 = \begin{pmatrix} W_2(-\partial^2 + \alpha^2) + W_3 & 0 \\ W_4 & W_2 \end{pmatrix}$$

then $\mathcal{V}_2\mathcal{V}_1 = \mathcal{V}$ and $\mathcal{Q}(\lambda) = \mathcal{V}_1\mathcal{L}(\lambda)^{-1}\mathcal{V}_2,\ \lambda \in \Omega$ is HS with kernel

$$\mathcal{Q}_\lambda(x,y) = \begin{pmatrix} W_3(x) & 0 \\ W_4(x) & W_2(x) \end{pmatrix} \mathcal{L}_\lambda(x,y) W_1(y) + W_2(x) \begin{pmatrix} q_{11}(x,y) & q_{12}(x,y) \\ 0 & 0 \end{pmatrix} W_1(y)$$

$$q_{11}(x,y) = \mathcal{G}_{22}(x-y) + \mathcal{G}_{22}(x+y) - \frac{2}{\mathcal{G}_{11}(0)}\,\mathcal{G}_{22}(x)\mathcal{G}_{11}(y)$$

$$q_{12}(x,y) = r_\lambda(x-y) - r_\lambda(x+y) + \frac{2}{\mathcal{G}_{11}(0)}\,\mathcal{G}_{22}(x)\mathcal{G}_{12}(y)$$

$$r_\lambda = (-\partial^2 + \alpha^2)\mathcal{G}_{12}$$

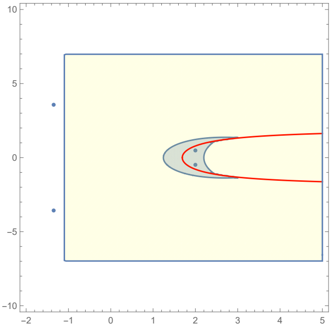
Ekman spectral problem - some challenges

- norm of BS-operator not decaying

$$\|\mathcal{Q}(\lambda)\| = \mathcal{O}(1), \quad \lambda \to \infty \text{ in } \Omega$$

- use spectral radius and similarity transform instead

$$\mathrm{r}(\mathcal{Q}(\lambda)) \leq \|\widetilde{\mathcal{Q}}(\lambda)\| = \mathcal{O}(|\lambda|^{-\frac{1}{2}}), \quad \lambda \to \infty \text{ in } \Omega$$



$$\|V\|_{L^1}^2 |\lambda|^{-\frac{1}{2}} \geq 1 \quad \text{vs.} \quad \alpha R\, \mathrm{r}(\mathcal{Q}(\lambda)) \geq 1$$

- estimate $\mathrm{r}(\mathcal{Q}(\lambda))$ by $\|\widetilde{\mathcal{Q}}(\lambda)\|_{\mathrm{HS}}$
- simple explicit bound in terms of α, R and L^1-norms of coefficients blows up around B_α

Thank you for your attention!