Non–self–adjoint relativistic point interactions and their approximations by non–local potentials

Author:
Bc. Lukáš Heriban

Supervisor:
Ing. Matěj Tušek, Ph.D.

Faculty of Nuclear Science and Physical Engineering CTU

Department of Mathematics
Program: Applied Algebra and Analysis

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Dirac operator

$$
\begin{aligned}
H_m &= -i\frac{\mathrm{d}}{\mathrm{d}x} \otimes \sigma_1 + m \otimes \sigma_3 \\
Dom(H_m) &= W^{1,2}(\mathbb{R}) \otimes \mathbb{C}^2
\end{aligned} \tag{1}
$$

$$
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
$$

$$
R_z(x,y) = \frac{i}{2}(\mathbb{Z}(z) + \operatorname{sgn}(x-y)\sigma_1)\mathrm{e}^{ik(z)|x-y|} \tag{2}
$$

where

$$
\mathbb{Z}(z) = \begin{pmatrix} \zeta(z) & 0 \\ 0 & \zeta^{-1}(z) \end{pmatrix},
$$

$$
\zeta(z) = \frac{z+m}{k(z)} \text{ and } k(z) = \sqrt{z^2 - m^2}, \operatorname{Im} k(z) \geq 0.
$$

Point interactions for a Dirac operator

$$\dot{H}\psi = H_m\psi$$
$$Dom(\dot{H}) = \{\varphi \in Dom(H_m) \mid \varphi(0) = 0\}$$

- Self–adjoint extension of $\dot{H}$ [S. Benvegnu, L.Dabrowski 1994 [5]]

$$H^{\mathbb{A}}\psi = H_m\psi$$
$$\mathrm{Dom}(H^{\mathbb{A}}) = \{\varphi \in W^{1,2}(\mathbb{R}\backslash\{0\})\otimes\mathbb{C}^2 \mid (2i-\sigma_1\mathbb{A})\varphi(0+) = (2i+\sigma_1\mathbb{A})\varphi(0-)\} \quad (3)$$
$$\mathbb{A} = \mathbb{A}^*$$

In a distribution sense [R.J.Huges 1999 [7]]

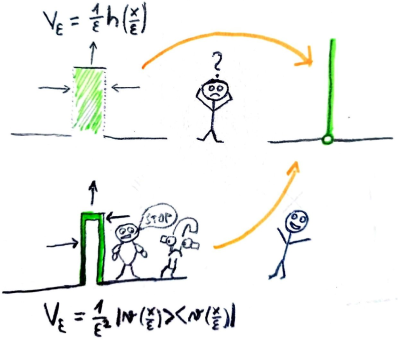
$$H^{\mathbb{A}}\psi = -i\frac{\mathrm{d}}{\mathrm{d}x}\otimes\sigma_1\psi + m\otimes\sigma_3\psi + \mathbb{A}\psi(0)\delta \quad (4)$$

Convergence of unbounded operators

Definition

Let $(A_n)_{n>0}$ and A be unbounded operators.

- *A_n converge to A in norm resolvent sense, if for all $z \in \rho(\mathbb{A})$, $\forall n \in \mathbb{N}, z \in \rho(\mathbb{A}_n)$ and resolvent $R_{A_n}(z)$ converge in operator norm to the resolvent $R_A(z)$.*



Approximations of relativistic point interactions

$$H_\varepsilon^{\mathbb{A}} = H_m + V_\varepsilon \otimes \mathbb{A}, \ \mathbb{A} = \mathbb{A}^*$$

$$V_\varepsilon \overset{\varepsilon \to 0+}{\longrightarrow} \delta$$

$$V_\varepsilon = \frac{1}{\varepsilon} v(x/\varepsilon)$$

Renormalization

[Hughes 1997,1999 [6, 7], Tusek 2019 [4]]

$$V_\varepsilon = \frac{1}{\varepsilon^2} |v(x/\varepsilon)\rangle\langle v(x/\varepsilon)|$$

Without renormalization

[L.Heriban 2020 [1]]

$$v \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}), \int_{\mathbb{R}} v = 1$$

Formal limits

$$H^{\mathbb{A}}\psi = -i\frac{\mathrm{d}}{\mathrm{d}x} \otimes \sigma_1\psi + m \otimes \sigma_3\psi + \mathbb{A}\psi(0)\delta \tag{5}$$

Generalization of the result

$$H_\varepsilon^{\mathbb{A}} = H_m + \frac{1}{\varepsilon^2}|v(x/\varepsilon)\rangle\langle v(x/\varepsilon)| \otimes \mathbb{A}$$

$$\mathbb{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ a, b, c, d \in \mathbb{C}$$

Formal limits

$$H^{\mathbb{A}}\psi = -i\frac{\mathrm{d}}{\mathrm{d}x} \otimes \sigma_1\psi + m \otimes \sigma_3\psi + \mathbb{A}\psi(0)\delta \tag{6}$$

Definition (General relativistic point interactions)

$$H^{\mathbb{A}}\psi = H_m\psi, \mathbb{A} \in \mathbb{C}^{2,2}$$

$$\mathrm{Dom}(H^{\mathbb{A}}) = \{\varphi \in W^{1,2}(\mathbb{R} \setminus \{0\}) \otimes \mathbb{C}^2 \mid (2i - \sigma_1\mathbb{A})\varphi(0+) = (2i + \sigma_1\mathbb{A})\varphi(0-)\}$$

Theorem

Let the matrix $\mathbb{A}$ in the definition of the Dirac operator with the non–local potential be any complex matrix and $z \in \mathbb{C} \setminus \{(-\infty, -m] \cup [m, +\infty)\}$ such that matrix

$$(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))$$

is invertible. Then the resolvent of the non–local potential converge in the operator norm to the bounded integral operator

$$R_z^{\mathbb{A}}(x, y) = R_z(x, y) - R_z(x, 0)(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}R_z(0, y). \qquad (7)$$

From resolvent formula we obtain resolvent of $H_\varepsilon^{\mathbb{A}}$ in the form of bounded integral operator

$$R_{z,\varepsilon}^{\mathbb{A}}(x,y) = \\ = R_z(x,y) - \int_{\mathbb{R}^2} R_z(x,\varepsilon t_1)(I + \langle v_\varepsilon | \mathbb{A} R_z v_\varepsilon \rangle)^{-1} |v\rangle\langle v|(t_1,t_2)\mathbb{A} R_z(\varepsilon t_2, y)\,\mathrm{d}t_1\,\mathrm{d}t_2. \tag{8}$$

Point–wise limit

$$R_{z,\varepsilon}^{\mathbb{A}} \overset{\varepsilon\to 0+}{\to} R_z^{\mathbb{A}} := R_z(x,y) - R_z(x,0)(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}R_z(0,y) \tag{9}$$

$$\|R_{z,\varepsilon}^{\mathbb{A}} - R_z^{\mathbb{A}}\|_{HS} \overset{\varepsilon\to 0+}{\to} 0$$

Theorem

Let the matrix $\mathbb{A}$ be any complex matrix and $z \in \mathbb{C} \setminus \{(-\infty, -m] \cup [m, +\infty)\}$ such that

$$(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))$$

is regular matrix. Then the operator

$$R_z^{\mathbb{A}} = R_z - R_z(x,0)(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}R_z(0,y) \tag{10}$$

is the resolvent of the operator $H^{\mathbb{A}}$.

Spectrum of relativistic point interactions

$$
\begin{aligned}
-i\sigma_1 \frac{\mathrm{d}}{\mathrm{d}x}\psi + m\sigma_3\psi = z\psi, \psi \in \mathrm{Dom}\, H^{\mathbb{A}} \\
\frac{\mathrm{d}}{\mathrm{d}x}\psi = i\begin{pmatrix} 0 & z+m \\ z-m & 0 \end{pmatrix}\psi.
\end{aligned} \tag{11}
$$

$$
\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} = \begin{pmatrix} \cos(k(z)x) & i\zeta(z)\sin(k(z)x) \\ i\zeta^{-1}(z)\sin(k(z)x) & \cos(k(z)x) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}
$$

$$
(2i\sigma_1 - \mathbb{A})\psi(0+) = (2i\sigma_1 + \mathbb{A})\psi(0-)
$$

Non–trivial solution if and only if

$$
0 = \det(2I + i\mathbb{A}\mathbb{Z}(z)) = 4 + 2i\mathrm{tr}(\mathbb{A}\mathbb{Z}(z)) - \det\mathbb{A}
$$

Theorem

Let $\mathbb{A}$ *be any complex matrix. Then*

$$\sigma(H^{\mathbb{A}}) \setminus \{(-\infty, -m] \cup [m, +\infty)\} = \sigma_p(H^{\mathbb{A}}) \setminus \{(-\infty, -m] \cup [m, +\infty)\}$$

and $z \in \mathbb{C} \setminus \{(-\infty, -m] \cup [m, +\infty)\}$ *is in the spectrum of the operator* $H^{\mathbb{A}}$ *if and only if* z *satisfies following equation*

$$4 + 2i\mathrm{tr}(\mathbb{A}\mathbb{Z}(z)) - \det \mathbb{A} = 0. \tag{12}$$

Spectral transition

If

$$\mathbb{A} = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}, b, c \in \mathbb{C}$$

$$0 = 4 + 2i\operatorname{tr}(\mathbb{A}\mathbb{Z}(z)) - \det \mathbb{A} = 4 - \det \mathbb{A}$$

- $\det \mathbb{A} \neq 4 \Rightarrow \sigma_p(H^{\mathbb{A}}) = \emptyset$
- $\det \mathbb{A} = 4 \Rightarrow \sigma(H^{\mathbb{A}}) = \mathbb{C}$

$$\mathbb{A} = \begin{pmatrix} 0 & (\delta+2)b \\ \frac{\delta-2}{b} & 0 \end{pmatrix}, \delta > 0, b \in \mathbb{R} \tag{13}$$

Theorem

For all $\varepsilon > 0$ and every $z \in \mathbb{C}$) there exists $\delta > 0$ such that $z \in \sigma_\varepsilon(H^{\mathbb{A}}) = \{z \in \mathbb{C} \mid \|R_z^{\mathbb{A}}\| > \varepsilon^{-1}\}$.

$$\|R^{\mathbb{A}}\| = \|R_z - R_z(x,0)(I + \tfrac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}R_z(0,y)\| \geq$$

$$\geq \frac{1}{\delta^2}\frac{1}{\sqrt{2}|z-m|^{1/4}|z+m|^{1/4}}\mathrm{e}^{-\frac{\sqrt{|z-m||z+m|}}{4}} - \frac{1}{\mathrm{dist}(z,\sigma(H_m))}$$

$$H_m = -i\hbar c \frac{d}{dx}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + mc^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$H_\varepsilon^{\mathbb{A}} = H_m + \frac{1}{\varepsilon^2}|v(x/\varepsilon)\rangle\langle v(x/\varepsilon)| \otimes \mathbb{A}$$

$H^{\mathbb{A}} = H_m$ on domain

$$\mathrm{Dom} H^{\mathbb{A}} = \{\psi \in W^{1,2}(\mathbb{R}\backslash\{0\}) \otimes \mathbb{C}^2 \mid (2i\sigma_1 + \mathbb{A})\psi(0-) = (2i\sigma_1 - \mathbb{A})\psi(0+)\}$$

$H_\varepsilon^{\mathbb{A}} \overset{NR}{\to} H^{\mathbb{A}}$ without renormalization

Spectrum: $z \in \sigma_p(H^{\mathbb{A}}) \leftrightarrow 0 = 4 + 2i\mathrm{tr}(\mathbb{A}\mathbb{Z}(z)) - \det \mathbb{A}$

Thank you for your attention

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[5] S. Benvegnu, L. Dabrowski *Relativistic point interaction in one dimension.* Letters in Mathematical Physics 30, 1994, 159-167.

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