The Möbius Strip

Tomáš Kalvoda
KAM FIT ČVUT

Herbertov
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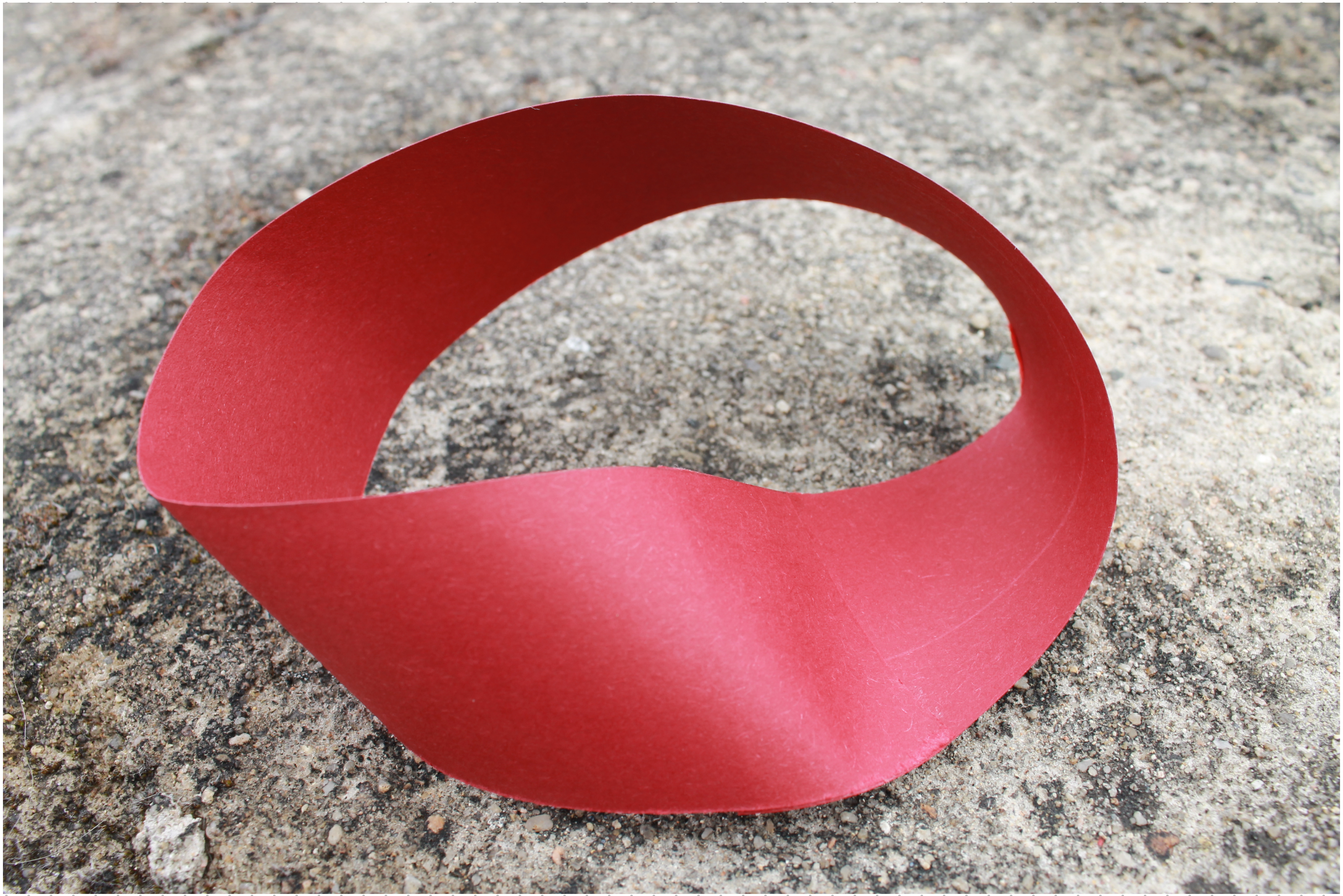


1/?

Contents

Augustus Ferdinand Möbius

- *1790 – † 1868

- studied astronomy under C.F. Gauss in Götingen
- spent majority of his life in Schulpforta, Saxony.
- analytical geometry, projective geometry, number theory, ...
- Möbius nets, function, plane, transformation, inversion formula, ...

The Möbius Strip

- the example of a non-orientable surface representable in $\mathbb{R}^3$.

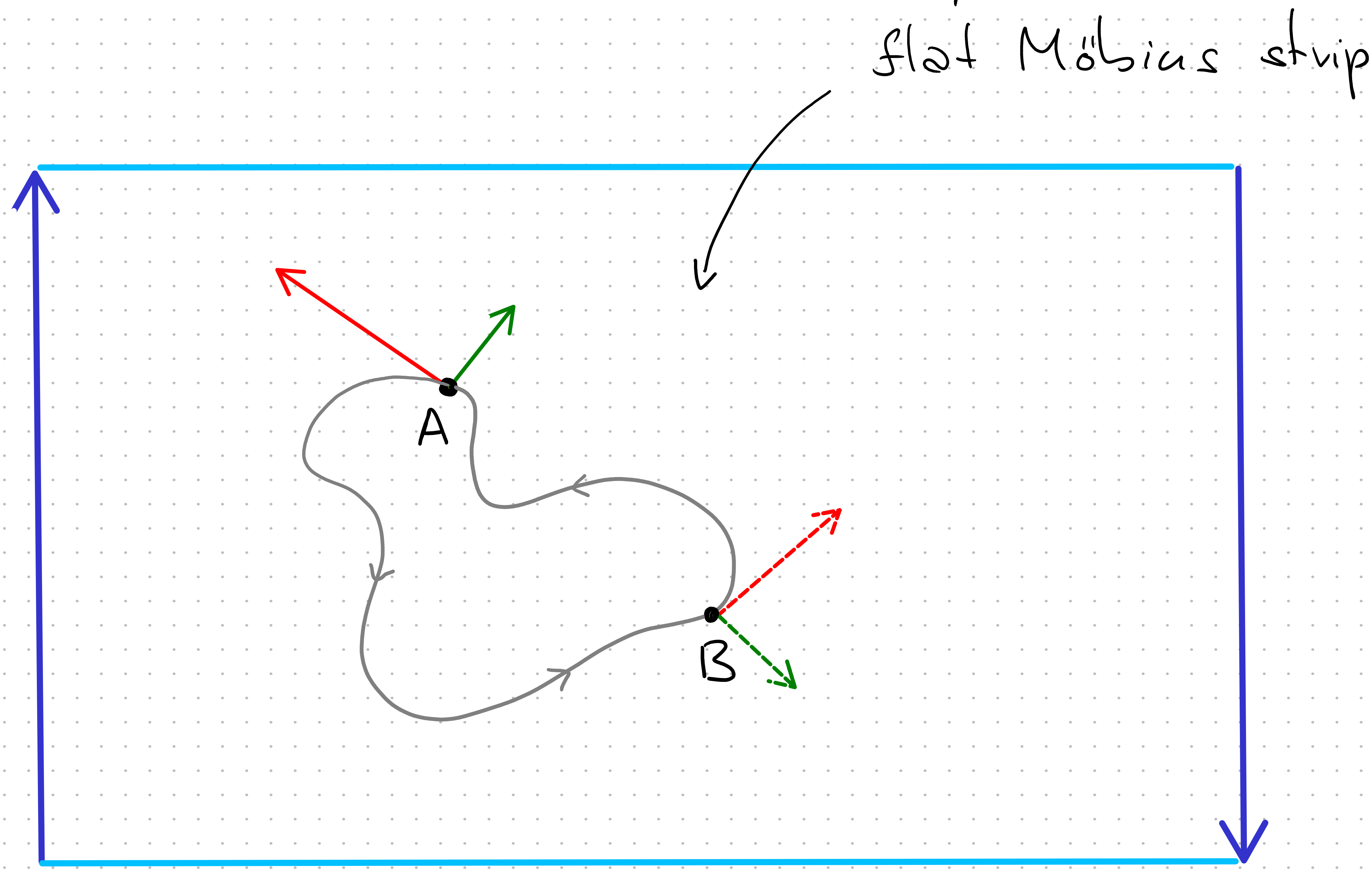
- discovered by Möbius in 1865 ("On the determination of the Volume of a Polyhedron").

- Johan Benedict Listing discovered the strip already in 1861.

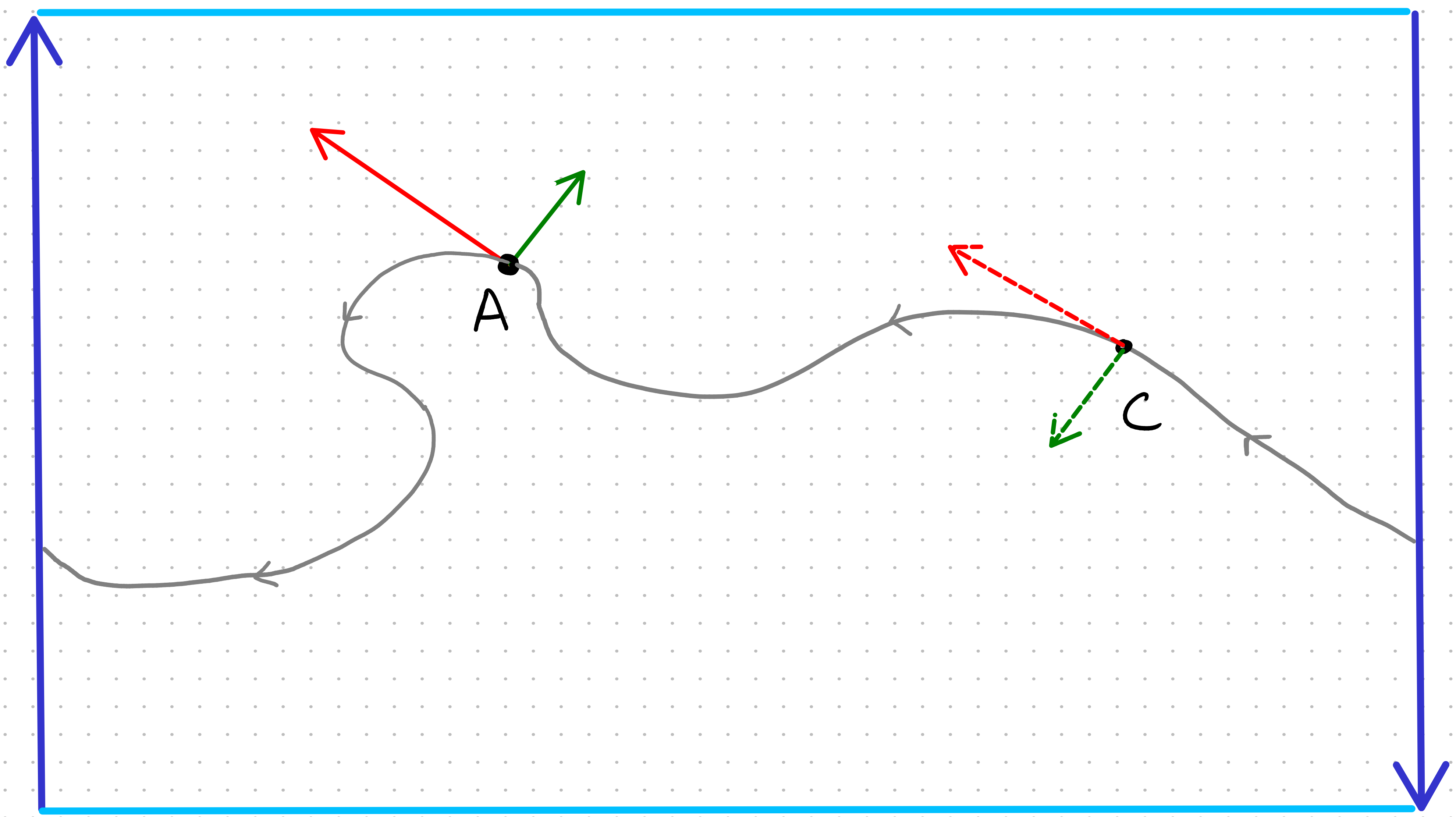
Arnold principle: If a notion bears a personal name, then this name is not the name of the discoverer.

- Möbius also explored the concept of "orientability" more deeply and his name stuck.

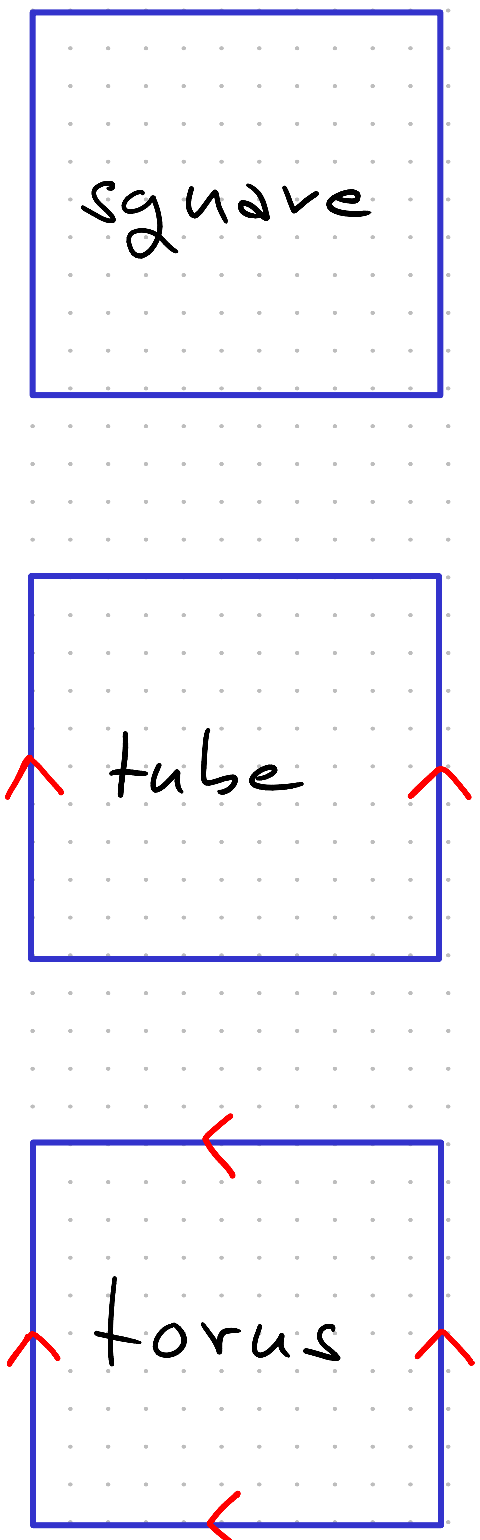
The flat version & non-orientability

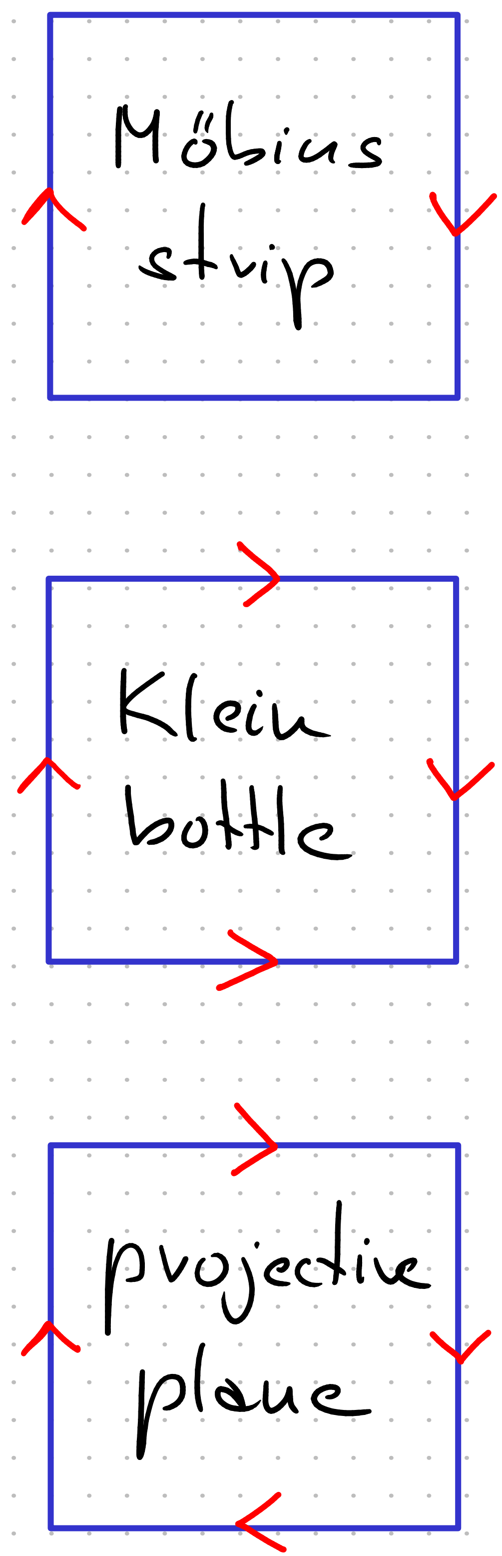


The flat version & non-orientability



Other orientable and non-orientable surfaces

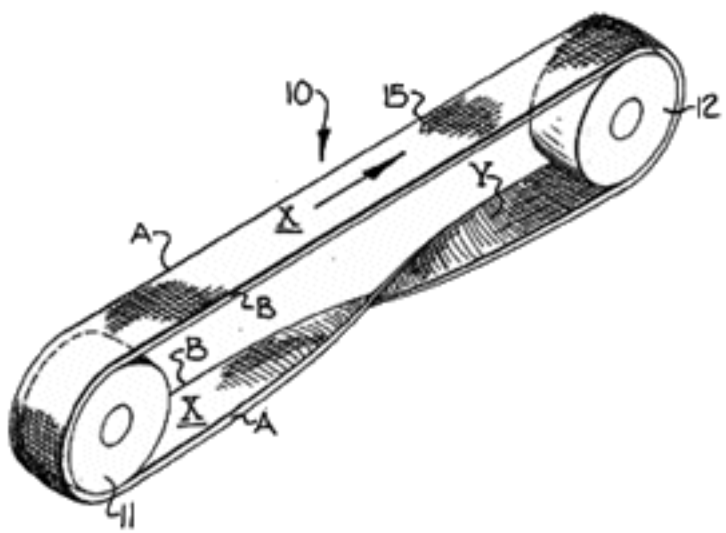




impossible with glue and paper

Are there any practical uses outside of topology textbooks?

- "magic tricks" (apply scissors), toys/puzzles
- conveyor belts & abrasive belts (old patents)



- electrical components (Möbius resistor & capacitator)
- synthesis of Möbius molecular band (~1980)

The "Recycle" symbol is a Möbius strip!

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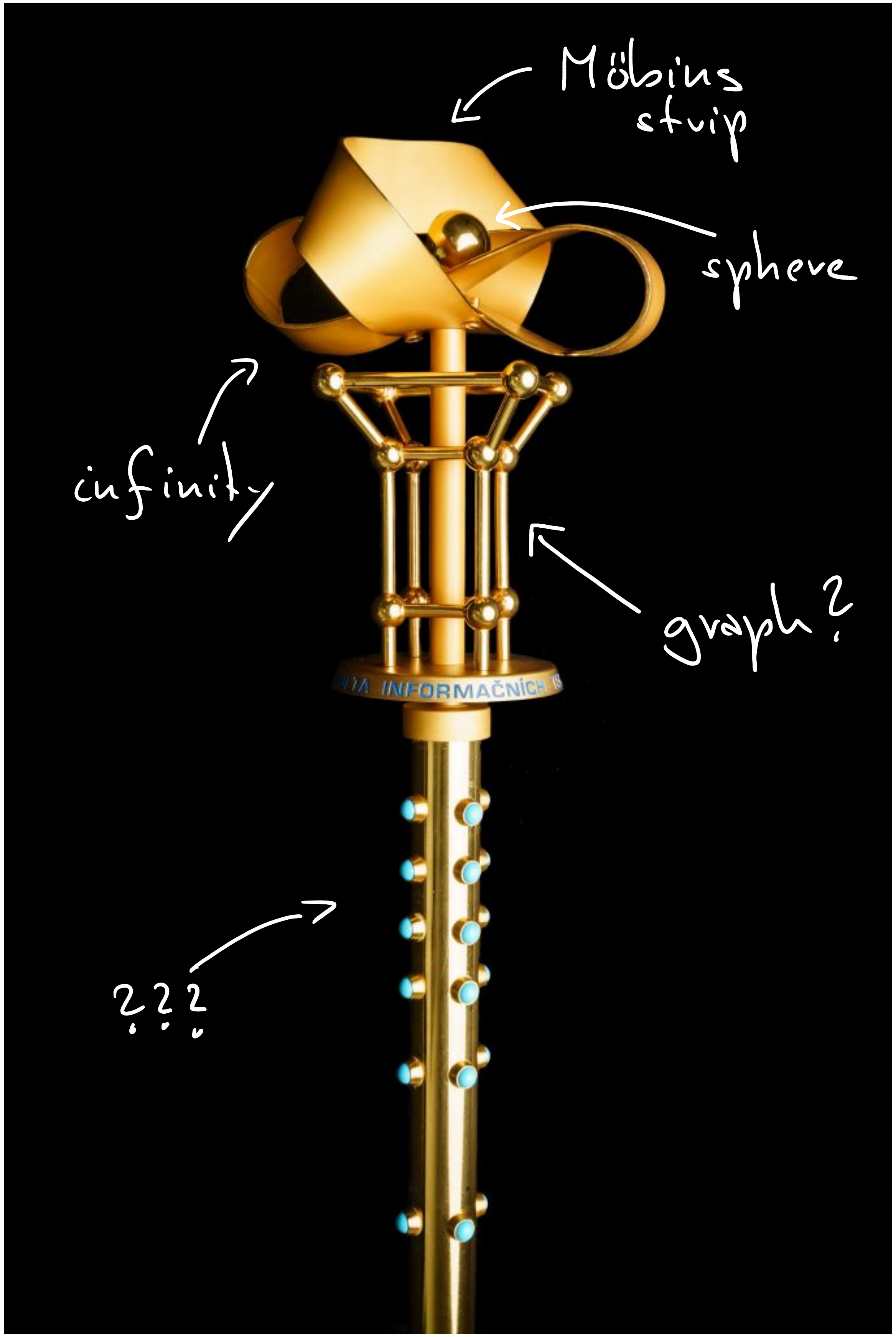
The scepter of the Faculty of Information Technology contains a Möbius strip!



Another Möbius' mathematical contribution:

- The Möbius function $\mu : \mathbb{Z} \longrightarrow \{-1, 0, 1\}$:

$$\mu(k) = \begin{cases} 0, & \underline{k} \text{ is divisible by a prime power,} \\ 1, & \underline{k} \text{ has even number of prime factors} \\ -1, & \text{— " — odd — " —} \end{cases}$$

$$\boxed{\sum_{k=1}^{\infty} \frac{\mu(k)}{k^s} = \frac{1}{\zeta(s)}}$$

- "Möbidromes"

$k \in \mathbb{N}$ is a Möbidrome $\overset{\text{def}}{\iff}$ base 10 digits of $\underline{k}$ form a palindrome and $\mu(k)$ is equal to all $\mu(\ell)$ where $\underline{\ell}$ runs through all right truncations of $\underline{k}$.

e.g.: $\mu(626) = 1$, $\mu(62) = 1$, $\mu(6) = 1$

$\mu(797) = -1$, $\mu(79) = -1$, $\mu(7) = -1$

"Möbidromes" (cont'd)

- $\mu(k) = 0$ are "trivial": $\mu(444\ldots4) = \mu(999\ldots9) = 0$.

- Pickover mentions $k = 79\ 737\ 873\ 797$ as the largest one known (2006).
  $\mu(k) = -1$, 11 digits.

- I have found:

  $\ell = 7\ 973\ 585\ 978\ 795\ 853\ 797$

  $\mu(\ell) = -1$, 19 digits

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that's enough of Math obscurities!

Part II:

Quantum Mechanics on the Möbius Strip

[T.K., D. Krejčiřík, K. Zahradová, 2020]

The Fake (or flat) Model

- Hilbert space: $L^2(\Omega_0, ds\,dt)$, $\Omega_0 = (0, 2\pi R) \times (-a, a)$
- Hamiltonian: $H_{flat} = -\Delta^{\Omega_0}$ ← the Laplacian

  with Dirichlet b.c.: $\varphi(s, \pm a) = 0$

  and twisted periodic b.c.: $\varphi(0, t) = \varphi(2\pi R, -t)$

  $$\partial_1 \varphi(0, t) = \partial_1 \varphi(2\pi R, -t)$$

- pros: the spectrum is explicitly computable,

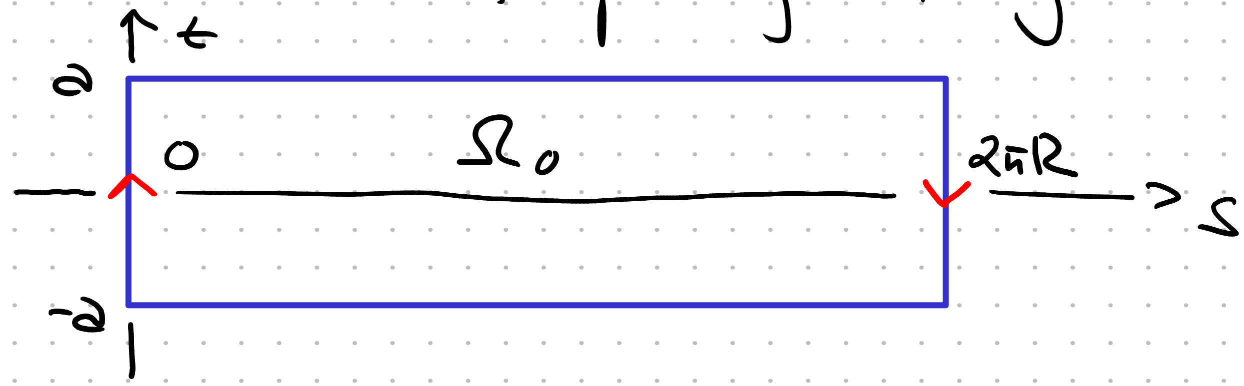
  $$\left\{ \left(\frac{m}{2R}\right)^2 + \left(\frac{n\pi}{2a}\right)^2 : m \in \mathbb{Z},\ n \in \mathbb{N}^*,\ m+n \text{ odd} \right\},$$

  eigenfunctions in terms of trigonometric functions.

- cons: influence of the geometry lost?

The True model

Let Ω be the Möbius strip of "length" $2\pi R$ and "width" $2a$:



$$\Omega = \left\{ \left(\left(R - t\cos\frac{s}{2R}\right)\cos\frac{s}{R},\ \left(R - t\cos\frac{s}{2R}\right)\sin\frac{s}{R},\ -t\sin\frac{s}{2R} \right) : s \in [0, 2\pi R),\ t \in (-a, a) \right\} \subset \mathbb{R}^3$$

and Hamiltonian:

$$H_{true} = -\Delta_D^{\Omega} \quad \text{in} \quad L^2(\Omega)$$

The Laplace-Beltrami operator in Ω with Dirichlet b.c. on $\partial\Omega$.

The True model (cont'd)

- the operator H_{true} is unitarily equivalent to

$$\tilde{H}_{true} = -\partial_1 \frac{1}{f_a^2} \partial_1 - \frac{1}{a^2} \partial_2^2 + V_a \text{ in } L^2((0,2\pi R) \times (-1,1))$$

with Dirichlet + twisted periodic b.c.,
where $f_a(s,u) = \left(\left(1 - \frac{au}{R} \cos \frac{s}{2R}\right)^2 + \left(\frac{au}{2R}\right)^2\right)^{1/2}$
and V_a is a "complicated" expression involving f_a and its derivatives.

- there is no hope to solve the spectral problem explicitly.

The effective Hamiltonian ($a \to 0_+$)

- we explore "thin" Möbius strips
- Let $E_1(a) = \left(\frac{\pi}{2a}\right)^2$ denotes the lowest transverse energy.
- Then we prove norm resolvent convergence:

$$\left\| U(H_{true} - E_1(a) - z)U^{-1} - (H_{eff} - E_1(a) - z)^{-1} \right\| = \\ = O(a),$$

where H_{eff} is the "effective" (not-so-fake) Hamiltonian (wait for next slide).

The not-so-fake model

- Hamiltonian in $L^2(\Omega_0)$:

$$H_{eff} = -\Delta_D^{\Omega_0} + V_{eff}, \quad V_{eff}(s,t) = -\frac{1}{8R^2}\cos\left(\frac{s}{R}\right)$$

+ Dirichlet and twisted periodic b.c.

- the spectrum:

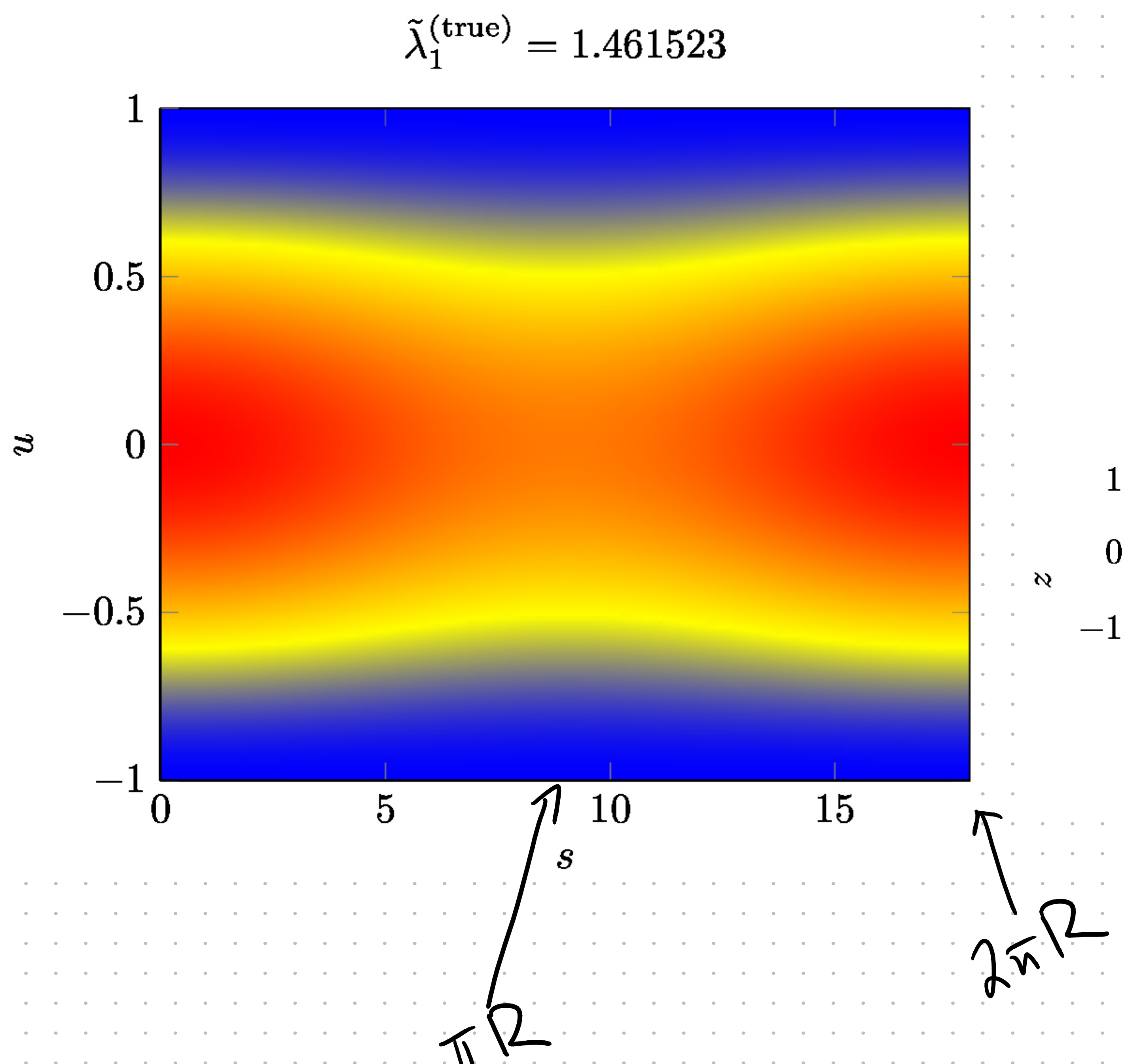
$$\sigma(H_{eff}) = \left\{ \left(\frac{1}{2R}\right)^2 a_m\left(-\frac{1}{4}\right) + \left(\frac{n\pi}{2a}\right)^2 : \begin{array}{l} m \in \mathbb{N},\ n \in \mathbb{N}^*, \\ m+n \text{ odd} \end{array} \right\}$$

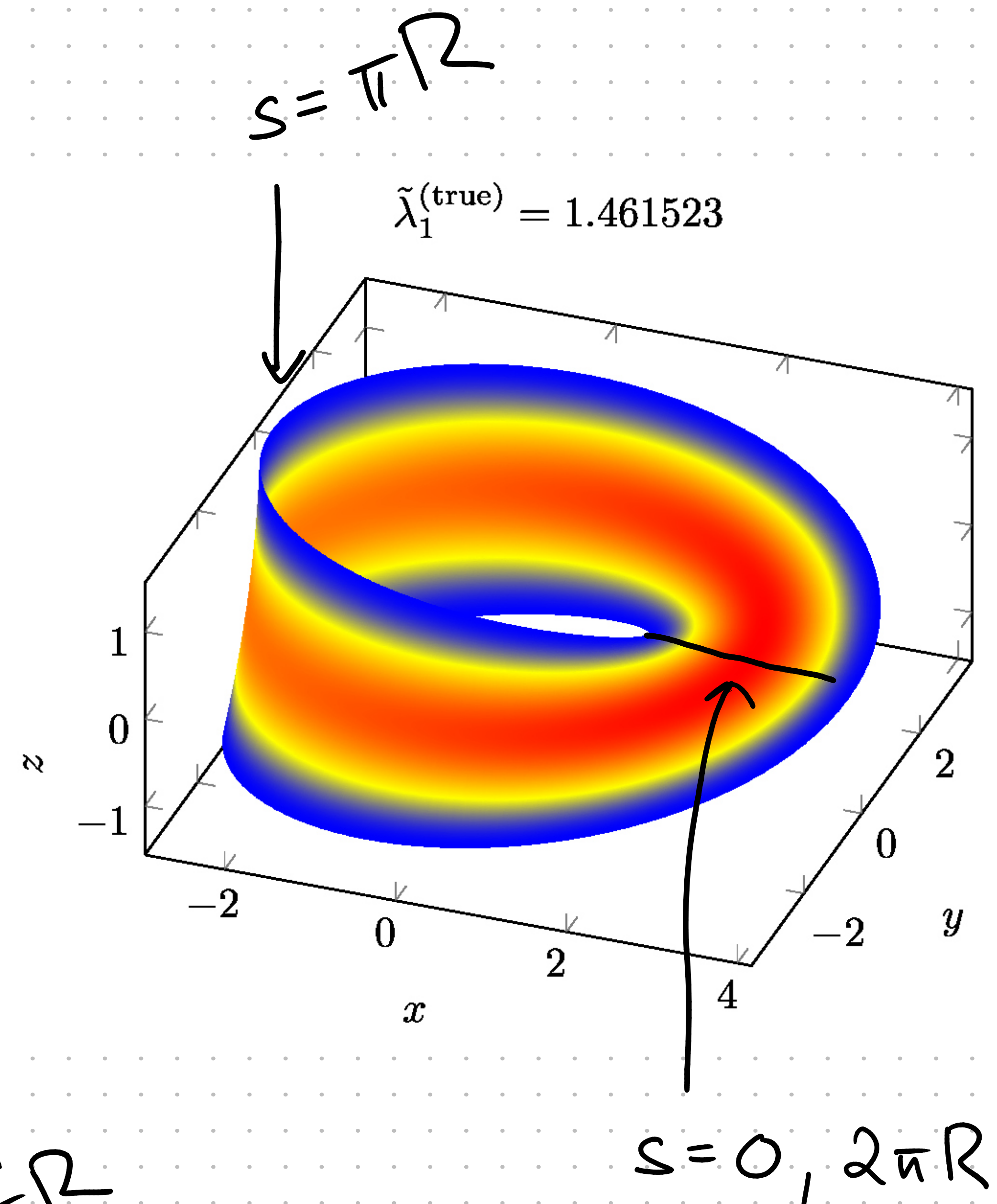
$$\cup \left\{ \left(\frac{1}{2R}\right)^2 b_m\left(-\frac{1}{4}\right) + \left(\frac{n\pi}{2a}\right)^2 : \begin{array}{l} m \in \mathbb{N}^*,\ n \in \mathbb{N}^*, \\ m+n \text{ odd} \end{array} \right\}$$

a_m, b_m are Mathieu characteristic values.

- eigenfunctions are expressible using Mathieu functions.

An Example ($a = 1{,}3$, $R = 18/2\pi$)





Thank you for your attention!

- T.K., D. Krejčiřík, K. Zahradová, Effective Quantum Dynamics on the Möbius Strip, J. Phys. A 53 (2020)
- C. Pickover, The Möbius Strip, Avalon, 2006