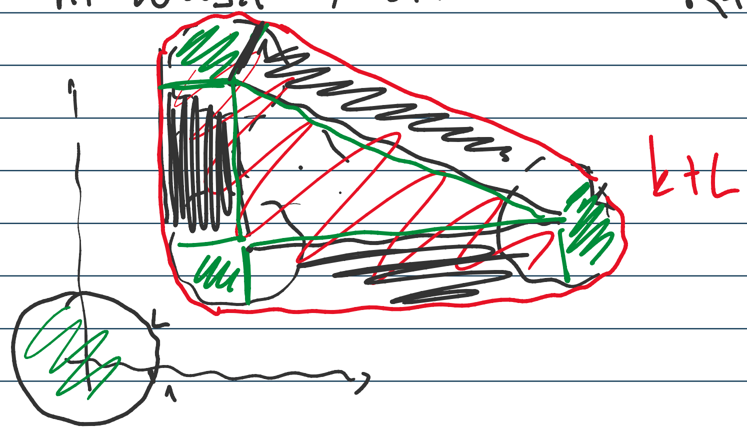
MIXED VOLUMES I.

- $\mathbb{R}^n$, vol = Lebesgue measure
- def: CONVEX BODY $K \subset \mathbb{R}^n$, $K \neq \emptyset$, COMPACT, CONVEX
  - SCALING λK, $\lambda \geq 0$
  - REFLECTION $-K$
  - TRANSLATION $K + x$, $x \in \mathbb{R}^n$
  - MINKOWSKI ADDITION : $K+L = \{k+l \,;\, k \in K, l \in L\}$



$$\boxed{2V(K,B) = \text{per}(K)}$$

- remark: Q on $\mathbb{R}^n$ $\rightsquigarrow$ $\exists!$ B - symmetric : $B(x,x) = Q(x)$

$$B(x,y) = \frac{1}{2}\frac{\partial^2}{\partial\alpha\,\partial\beta} Q(\alpha x + \beta y)$$

(Thm) (MINKOWSKI, 1911) $K_1, \ldots K_m \subset \mathbb{R}^n$ convex bodies

Then $\text{vol}(\lambda_1 K_1 + \lambda_2 K_2 + \cdots \lambda_m K_m)$

is a pol. of deg. n in $\lambda_1 \dots \lambda_n \geq 0$.

def: MIXED VOLUME:

$$V(K_1, \dots K_n) = \frac{1}{n!} \frac{\partial^n}{\partial \lambda_1 \dots \partial \lambda_n} \mathrm{vol}(\lambda_1 K_1 + \dots \lambda_n K_n).$$

rmk: $\boxed{n=2}$ $2\,V(K,L) = \mathrm{vol}(K+L) - \mathrm{vol}(K) - \mathrm{vol}(L)$

PROPERTIES of V:

a) $V(K, \dots K) = \mathrm{vol}(K)$

b) SYMMETRIC

c) $V(\lambda K_1, \dots, K_n) = \lambda V(K_1 \dots K_n) \quad \forall \lambda \geq 0$

d) $V(K+L, K_2 \dots K_n) = V(K, K_2 \dots) + V(L, K_2 \dots K_n)$

e) $V(K+x, K_2, \dots) = V(K, K_2 \dots)$

f) $V(L_j, K_2 \dots) \to V(L, K_2 \dots)$ if $L_j \to L$

g) $V(K \cup L, K_2 \dots K_n) = V(K, K_2 \dots) + V(L, K_2 \dots) - V(K \cap L, K_2 \dots)$

V - a transl. inv. cont. valuation

h) $V(K_1 \dots K_n) \geq 0$

i. $V(K, K_2 \dots K_n) \leq V(L, K_2 \dots K_n)$ if $K \subset L$

i, $V(K_1, K_2, \ldots, K_n) \leq V(L_1, K_2, \ldots, K_n)$ if $K \subset L$

· THM ALEXANDROV-FENCHEL INEQ.

(ALEXANDROV, 1937, 1938)

$$\boxed{V(K, L, K_3, \ldots, K_n)^2 \geq V(K, K, K_3, \ldots, K_n)\, V(L, L, K_3, \ldots, K_n)}$$

RM $n=2$: $L = B$ $\quad V(K, B)^2 \geq V(K, K) \cdot V(B, B)$

$$\frac{1}{4} \cdot \mathrm{per}(K)^2 \geq \mathrm{vol}(K) \cdot \pi$$

$$\boxed{\mathrm{vol}(K) \leq \frac{\mathrm{per}(K)^2}{4\pi}}$$

REF BOOKS : GRUBER

BURAGO-ZALLGALLER

SCHNEIDER

MIXED VOLUMES II.

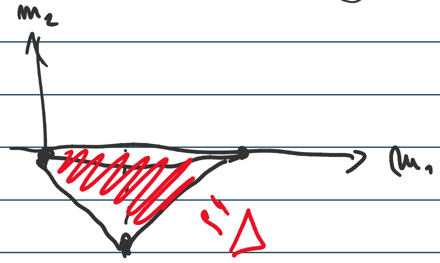
· def: NEWTON POLYTOPE of a Laurent pol. $\quad z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}$

def: Newton polytope of a Laurent pol.

Δ

$$z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n} \longmapsto (m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n \subset \mathbb{R}^n$$

(Ex) $P(z_1,z_2) = \alpha \cdot \frac{z_1}{z_2^2} + \beta \cdot z_1^2 + \gamma$



(Thm) (D.N. Bernstein, 1976)

The number of roots $P_1 = P_2 = \cdots = P_n = 0$ is

$$n!\, V(\Delta_1, \ldots, \Delta_n).$$

New proofs of AF:

a) Khovanskii, Teissier, 1978

b) McMullen (1993, Inventiones)

· Minkowski sum of polytopes $\rightsquigarrow$ algebra

$\rightsquigarrow$ graded algebra. $A = \bigoplus_{k=0}^{n} A_k$, $A_k \cdot A_l \subset A_{k+l}$

$A_0 = A_n \cong \mathbb{R}$

(PD) $A_k \times A_{n-k} \to \mathbb{R}$

$(a, b) \longmapsto a \cdot b$

(HL) $\omega \in A_1$: $A_k \xrightarrow{\sim} A_{n-k}$

$a \mapsto a \cdot \omega^{n-2k}$

(HR) SIGNATURE OF $Q(a,b) = a \cdot b \cdot \omega^{n-2k}$

$k \le \frac{n}{2}$ $\qquad a, b \in A_k$

$\Rightarrow$ $\underline{k=0}$: $V \geq 0$

$\underline{k=1}$: AF

$\vdots$

?

cf [illegible] – [illegible] (ANNALS, 2012)

d, 2018, 2019,

VALUATIONS

- MCMULLEN (1980) : M.V. span a dense subset in Val
- ALESKER (2001) : $\boxed{\text{IRREDUCIBILITY THEOREM}}$ $\rightarrow$ ETS

$\boxed{\text{PRODUCT}}$ on Val

PRODUCT ON ...

$$\int V(\cdot,\ldots,\cdot, A_1,\ldots,A_k) * V(\ldots, B_1,\ldots,B_k) = \text{cst}\, V(\ldots, A \ldots A, B, \ldots B)$$

k

(PD) ALESKER, 2003

(HL) ALESKER, BERNIG + BRÖCKER 2007 – 2011

(HR) k, k – LAMBDA...

→ $k = 0$: $V \geq 0$

→ $k = 1$: AF

• EQUALITY CASES

• POLYTOPES — 1978 (SCHNEIDER) CONJ.
VAN HANDEL, SHENFELD (2020)

IN GENERAL → OPEN

• GODBERSON'S CONJ. (1938)

$$V(\underbrace{K, K, \ldots, K}_{k\text{-times}}, -K, \ldots, -K) \leq \binom{n}{k} \operatorname{vol}(K).$$