Solvable models in quasi-Hermitian quantum mechanics

David Kramár
Supervisor: doc. Mgr. David Krejčiřík Ph.D. DSc.

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SVK - Herbertov

$\mathcal{PT}$-symmetric QM

Definition ($\mathcal{PT}$-symmetry)

Let $\mathcal{H} := L^2(\mathbb{R}^n)$ be the Hilbert space, H arbitrary operator on $\mathcal{H}$. We say H is $\mathcal{PT}$-symmetric if it satisfies

$$[\mathcal{PT}, H] = 0,$$

where $\mathcal{P}\psi(x) := \psi(-x)$ and $\mathcal{T}\psi(x) := \overline{\psi}(x)$, for all $\psi \in \mathcal{H}$ is the space-reversal and the time-reversal operator, respectively.

- Carl M. Bender and Stefan Boettcher, 1998

$$-\frac{d^2}{dx^2} + ix^3 \qquad \text{in} \qquad L^2(\mathbb{R})$$

- Ali Mostafazadeh, 2002

Quasi-Hermitian QM

Definition (Metric operator)

We say positive bounded operator Θ with bounded inverse is a Metric operator for the given operator H on the space $\mathcal{H}$ if it satisfies

$$\Theta H = H^* \Theta.$$

Such operator is then called quasi-Hermitian or quasi-self-adjoint.

- F. G. Scholtz, H. B. Geyer and F. J. W. Hahne, 1992
- For every $\Omega \in \mathcal{B}(\mathcal{H})$ such that $\Theta = \Omega^* \Omega$ the operator $h := \Omega H \Omega^{-1}$ is similar to H and self-adjoint.

Theorem

Let H be an operator with purely discrete spectrum. Then H is quasi-self-adjoint, if and only if the eigenvectors of its adjoint, denoted as $(\phi_n)_{n=0}^{\infty}$, form a Riesz basis.

Definition (Riesz basis)

A sequence $(\phi_n)_{n=0}^{\infty} \subset \mathcal{H}$ is said to be Riesz basis for $\mathcal{H}$ if it is an image of some orthonormal basis $(e_n)_{n=0}^{\infty}$ under bounded and boundedly invertible linear transformation T.

$$\Theta := \sum_{n=0}^{\infty} \phi_n \left(\phi_n, \cdot\right), \qquad \Omega := \sum_{n=0}^{\infty} e_n \left(\phi_n, \cdot\right).$$

The model H_α

Given any real positive number d, we consider the Hilbert space $\mathcal{H} := L^2(0,d)$ and for every $\alpha \in \mathbb{R}$ we define the operator H_α on the Hilbert space $\mathcal{H}$ as

$$H_\alpha \psi := -\psi'' \quad \text{in} \quad (0,d), \tag{1}$$

$$\psi'(0) + i\alpha\psi(0) = 0, \qquad \psi'(d) + i\alpha\psi(d) = 0, \tag{2}$$

with its operator domain

$$\mathrm{Dom}(H_\alpha) := \{\psi \in W^{2,2}(0,d) \mid \psi \text{ satisfies (2)}\}.$$

- Krejčiřík, Bíla and Znojil, 2006
- For the value $\alpha = 0$ and $\alpha = \infty$ we obtain (for $\alpha = \infty$ only formally) Neumann and Dirichlet Laplacian, respectively, i.e.

$$H_0 = -\Delta_N, \qquad H_\infty = -\Delta_D.$$

Theorem (Krejčiřík, 2008)

Metric operator for H_α exists, if and only if $\alpha \neq \frac{n\pi}{d}$ for all $n \in \mathbb{Z}$ and it is given as

$$\Theta_\alpha = \mathbb{I} + K_\alpha,$$

where K_α is a Hilbert-Schmidt integral operator with the kernel $\mathcal{K}_\alpha$ given by

$$\mathcal{K}_\alpha(x,y) = \phi_0^\alpha(x)\overline{\phi_0^\alpha}(y) - \frac{1}{d} + \alpha^2 \mathcal{G}_D(x,y) - i\alpha\left(\frac{y-x}{d} + \operatorname{sgn}(y-x)\right).$$

where $\mathcal{G}_D := \frac{x(d-y)}{d}$ for $0 < x < y < d$, with x, y exchanged for $x > y$.

Spectral properties

- Spectrum: $\sigma(H_\alpha) = \{\alpha^2\} \cup \{k_n^2 \mid n \in \mathbb{N}\}$ with $k_n := \frac{n\pi}{d}$.
- Eigenfunctions: $(H_\alpha - \lambda_n)\,\psi_n^\alpha = 0, \qquad (H_\alpha^* - \lambda_n)\,\phi_n^\alpha = 0$

$$\psi_0^\alpha(x) = A_0 \mathrm{e}^{-i\alpha x}, \qquad \psi_n^\alpha(x) = A_n\left(\psi_n^N(x) - i\frac{\alpha}{k_n}\psi_n^D(x)\right),$$

$$\phi_0^\alpha(x) = B_0 \mathrm{e}^{i\alpha x}, \qquad \phi_n^\alpha(x) = B_n\left(\psi_n^N(x) + i\frac{\alpha}{k_n}\psi_n^D(x)\right),$$

where

$$\psi_n^N(x) := \sqrt{\frac{2}{d}}\cos(k_n x), \qquad \psi_n^D(x) := \sqrt{\frac{2}{d}}\sin(k_n x).$$

- Generalized eigenfunctions:

$$(H_\alpha - \lambda_n)\,\xi_n^\alpha = \lambda_n \psi_n^\alpha : \qquad \xi_m^\alpha := A_0 \frac{1}{2k_m}\left(-\frac{1}{2k_m}\mathrm{e}^{ik_m x} + ix\mathrm{e}^{-ik_m x}\right),$$

$$(H_\alpha^* - \lambda_n)\,\eta_n^\alpha = \lambda_n \psi_n^\alpha : \qquad \eta_m^\alpha := B_0 \frac{1}{2k_m}\left(-\frac{1}{2k_m}\mathrm{e}^{-ik_m x} - ix\mathrm{e}^{ik_m x}\right).$$

- Exists only if $\alpha = k_m$ for some $m \in \mathbb{N}$.

Theorem (Krejčiřík, Siegl and Železný 2014)

For all $\alpha \in \mathbb{R}$, eigenfunctions of H_α^* together with generalized eigenfunctions of H_α^* form a Bari basis.

Definition (Bari basis)

A sequence $(\phi_n)_{n=0}^{\infty} \subset \mathcal{H}$ is said to be a Bari basis for $\mathcal{H}$ if there is an orthonormal basis $(e_n)_{n=0}^{\infty}$ such that

$$r^2 := \sum_{n=0}^{\infty} \|\phi_n - e_n\|^2 < \infty,$$

and for every complex sequence $(\alpha_n)_{n=0}^{\infty} \subset \mathbb{C}$

$$\sum_{n=0}^{\infty} \alpha_n \phi_n = 0 \Rightarrow \alpha_n = 0, \ \forall n \in \mathbb{N}_0.$$

Theorem (Krejčiřík, Siegl and Železný, 2014)

Let $\alpha \neq \frac{n\pi}{d}$ for every $n \in \mathbb{Z}$. Then there exists $\Omega \in \mathcal{B}(\mathcal{H})$ such that $\Omega^{-1} \in \mathcal{B}(\mathcal{H})$ and the transformed operator $h_\alpha := \Omega H_\alpha \Omega^{-1}$ satisfies

$$h_\alpha := H_0 + \alpha^2 \psi_0^N \left(\psi_0^N, \cdot\right) \qquad \text{with} \qquad \psi_0^N(x) := \sqrt{\frac{1}{d}}\,.$$

- $\Omega : \psi_n^\alpha \mapsto \psi_n^N : \sum_{n=0}^{\infty} \psi_n^N \left(\phi_n, \cdot\right)$

New results

Theorem (Similar self-adjoint operator corresponding to the Dirichlet basis)

Let $\alpha \neq k_n$ for every $n \in \mathbb{Z}$. Then there exists $\Omega_D \in \mathcal{B}(\mathcal{H})$ such that $\Omega_D^{-1} \in \mathcal{B}(\mathcal{H})$ and the transformed operator $h_\alpha^D := \Omega_D H_\alpha \Omega_D^{-1}$ satisfies

$$h_\alpha^D = \left[(-\Delta_D)^{\frac{1}{2}} - k_1\right]^2 + \alpha^2 \psi_1^D \left(\psi_1^D, \cdot\right).$$

New results

Theorem (Similar operator for the exceptional points)

Let $\alpha = k_n$ for some $n \in \mathbb{Z}$. Then there exist $\tilde{\Omega}, \tilde{\Omega}_D \in \mathcal{B}(\mathcal{H})$ such that $\tilde{\Omega}^{-1}, \tilde{\Omega}_D^{-1} \in \mathcal{B}(\mathcal{H})$ and the transformed operators $h_\alpha := \tilde{\Omega} H_\alpha \tilde{\Omega}^{-1}$ and $h_\alpha^D := \tilde{\Omega}_D H_\alpha \tilde{\Omega}_D^{-1}$ satisfy

$$h_\alpha = H_0 + \alpha^2 \psi_0^N \left[\left(\psi_0^N, \cdot \right) + \left(\psi_n^N, \cdot \right) \right]$$

$$h_\alpha^D = \left[(-\Delta_D)^{\frac{1}{2}} - k_1 \right]^2 + \alpha^2 \psi_1^D \left[\left(\psi_1^D, \cdot \right) + \left(\psi_n^D, \cdot \right) \right]$$

- For the Neumann orthonormal basis $\left(\psi_n^N\right)_{n=0}^{\infty}$ we define $\Omega := \sum_{n=0}^{\infty} \psi_n^N \left(\phi_n^\alpha, \cdot\right)$ with $\alpha = k_m$ for some $m \in \mathbb{N}$. Then

$$\Omega^{-1} = \sum_{n=2}^{\infty} \psi_{\sigma(n)}^\alpha \left(\psi_{\sigma(n)}^N, \cdot\right) + \psi_m^\alpha \left(\psi_0^N, \cdot\right) + \psi_0^\alpha \left(\psi_m^N, \cdot\right) - \psi_m^\alpha \left(\psi_m^N, \cdot\right),$$

$$\Omega^* = \sum_{n=0}^{\infty} \phi_n^\alpha \left(\psi_n^N, \cdot\right),$$

where $\sigma : \mathbb{N}_0 \mapsto \mathbb{N}_0$ is a transposition of elements 1 and m.

- Further we introduce the unitary operator

$$U := \sum_{n=0}^{\infty} \psi_{n+1}^D \left(\psi_n^N, \cdot\right) : \psi_n \mapsto \psi_{n+1}^D.$$

H_α in the Dirichlet basis

$$
\begin{aligned}
h_\alpha^D &= \Omega_D H_\alpha (\Omega_D)^{-1} = U \Omega_N H_\alpha (\Omega_N)^{-1} U^{-1} \\
&= U(-\Delta_N) U^{-1} + \alpha^2 U \psi_0^N \left(\psi_0^N, U^{-1} \, \cdot \right) \\
&= \sum_{n=0}^{\infty} k_n^2 \; \psi_{n+1}^D \left(\psi_{n+1}^D, \cdot \right) + \alpha^2 \psi_1^D \left(\psi_1^D, \cdot \right) \\
&= \sum_{n=1}^{\infty} k_{n-1}^2 \; \psi_n^D (\psi_n^D, \cdot) + \alpha^2 \psi_1^D \left(\psi_1^D, \cdot \right) \\
&= \sum_{n=1}^{\infty} k_n^2 \; \psi_n^D (\psi_n^D, \cdot) - 2k_1 \sum_{n=1}^{\infty} k_n \; \psi_n^D (\psi_n^D, \cdot) + k_1^2 \sum_{n=1}^{\infty} \psi_n^D (\psi_n^D, \cdot) \\
&\quad + \alpha^2 \psi_1^D \left(\psi_1^D, \cdot \right) \\
&= \left[(-\Delta_D)^{\frac{1}{2}} - k_1 \mathbb{I} \right]^2 + \alpha^2 \psi_1^D \left(\psi_1^D, \cdot \right)
\end{aligned}
$$

Degenerate spectrum

- Let $\alpha = k_m$ for some $m \in \mathbb{N}$, then for $\phi, \psi \in \mathrm{Dom}(-\Delta_N)$ we have

$$
\begin{aligned}
(\phi, h_\alpha \psi) &= (\phi, \tilde{\Omega} H_m \tilde{\Omega}^{-1} \psi) = \\
&= \lim_{l \to \infty} (H_m^* \tilde{\Omega}^* \phi, \sum_{n=2}^{l} \psi_{\sigma(n)}^{\alpha} \left(\psi_{\sigma(n)}^{N}, \psi \right) + \psi_m^{\alpha} \left(\psi_0^{N}, \psi \right) \\
&\quad + \psi_0^{\alpha} \left(\psi_m^{N}, \psi \right) - \psi_m^{\alpha} \left(\psi_m^{N}, \psi \right)) \\
&\ldots \\
&= (\phi, (-\Delta_N) \psi + k_m^2 \, \psi_0^N \left[\left(\psi_0^N, \psi \right) + \left(\psi_m^N, \psi \right) \right]).
\end{aligned}
$$