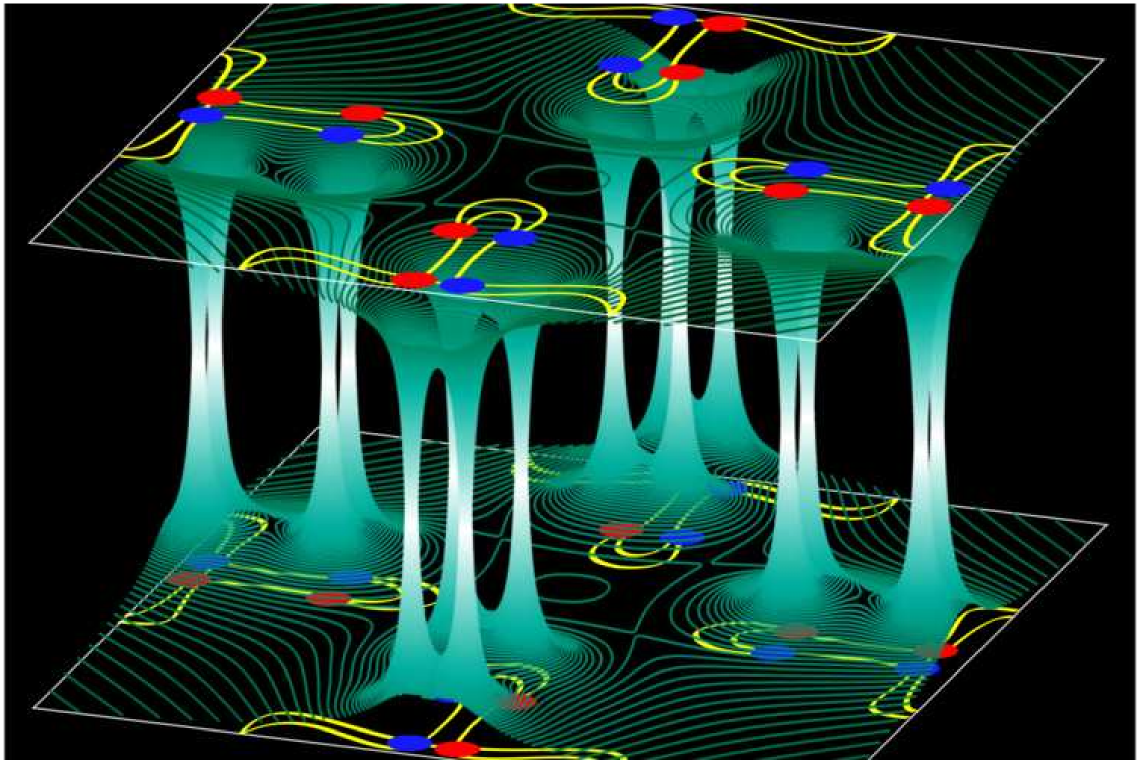
Bound states in semi-Dirac semi-metals

David KREJČIŘÍK

http://people.fjfi.cvut.cz/krejcirik

Czech Technical University in Prague



Yazdani Lab - Princeton University

Based on: Physics Letters A (2021) [with P. R. S. Antunes]

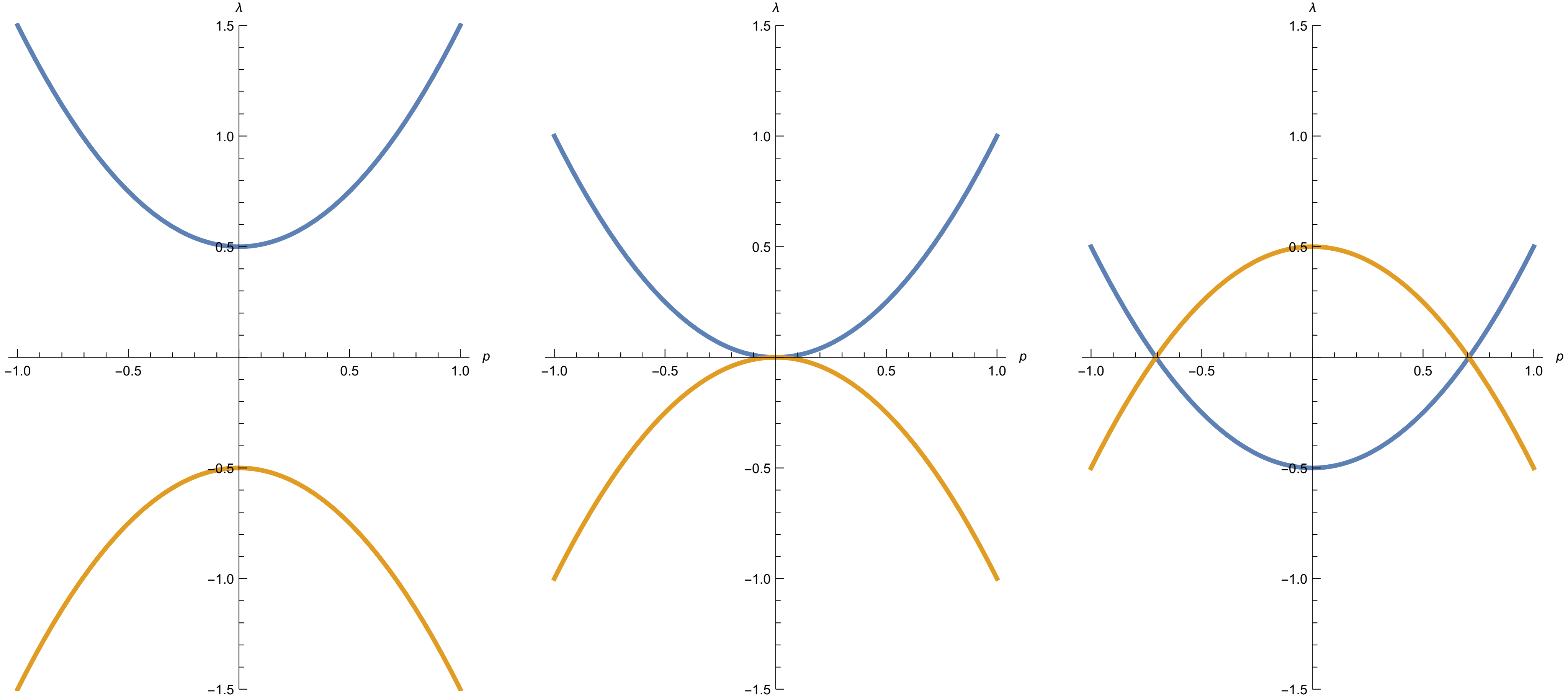
Conventional materials

$$H = \begin{pmatrix} 0 & p^2+\delta \\ p^2+\delta & 0 \end{pmatrix}$$ in $L^2(\mathbb{R};\mathbb{C}^2)$

$$\sigma(H) = \pm \bigcup_{p\in\mathbb{R}} (p^2+\delta)$$

quadratic dispersion

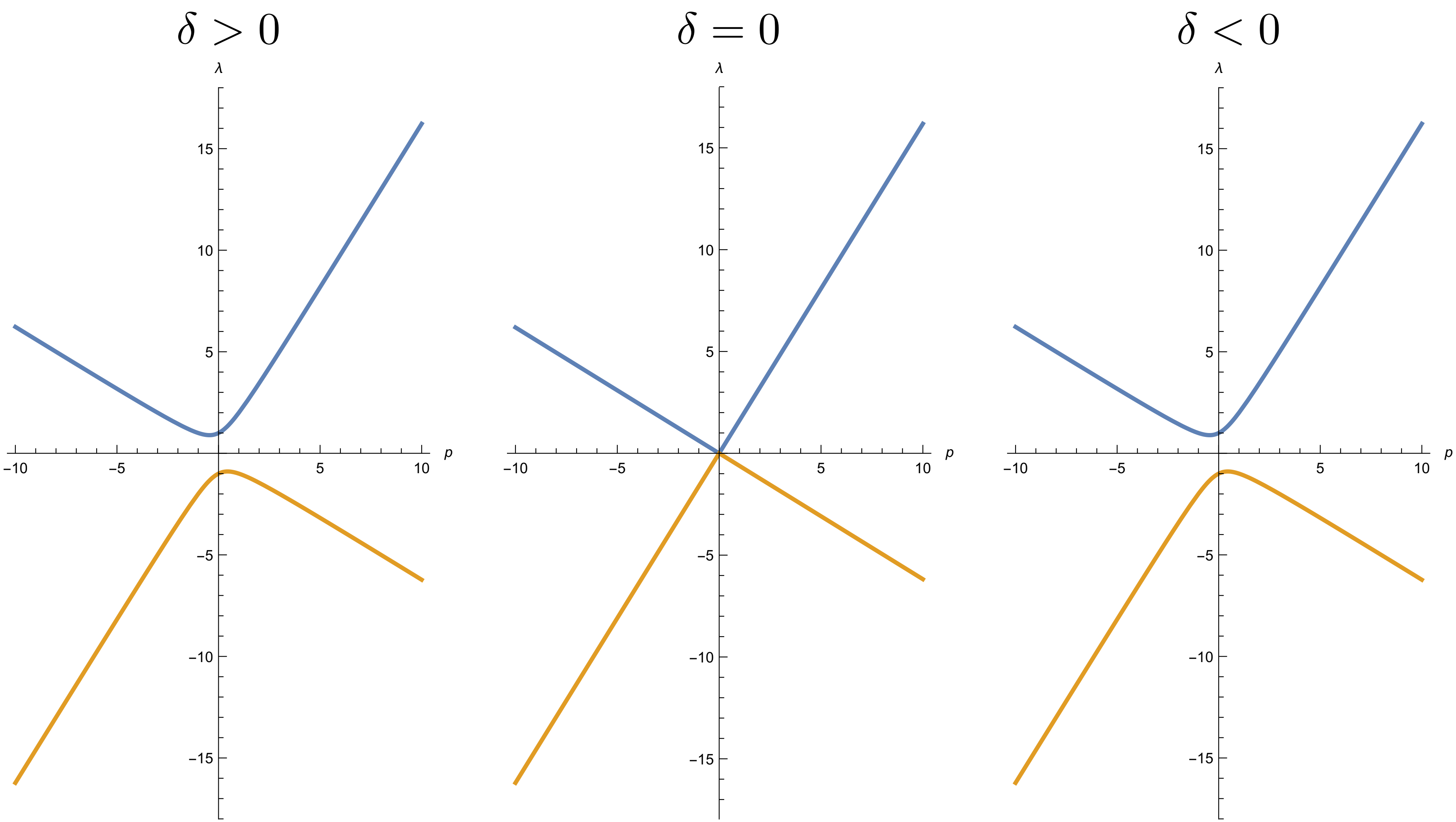
$\delta > 0$ $\delta = 0$ $\delta < 0$



Dirac materials

$$H = \begin{pmatrix} p & \delta \\ \delta & -p \end{pmatrix} \quad \text{in} \quad L^2(\mathbb{R};\mathbb{C}^2) \qquad \sigma(H) = \bigcup_{p\in\mathbb{R}} \frac{p \pm \sqrt{5p^2+4\delta^2}}{2}$$

linear dispersion



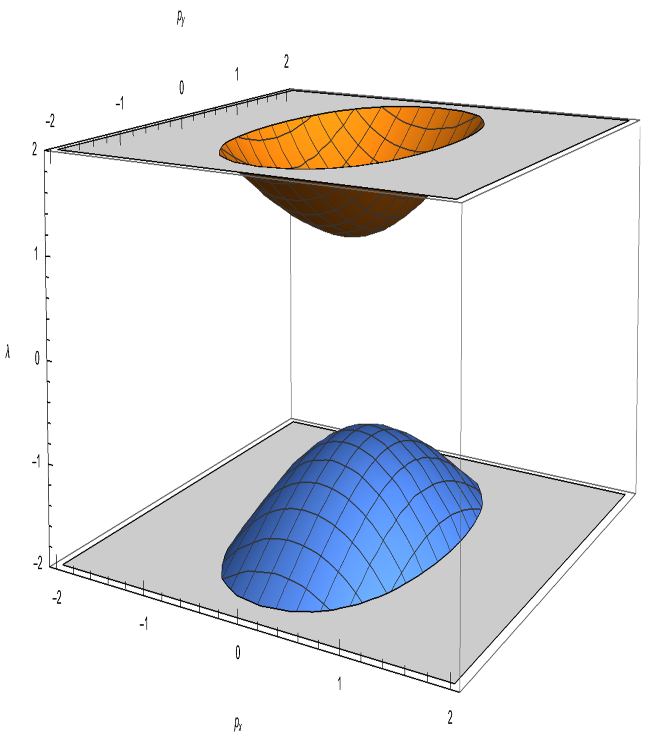
semi-Dirac semi-metals

$$H = \begin{pmatrix} p_y & p_x^2 + \delta \\ p_x^2 + \delta & -p_y \end{pmatrix}$$ in $L^2(\mathbb{R}^2; \mathbb{C}^2)$

$$\sigma(H) = \pm \bigcup_{p_x, p_y \in \mathbb{R}} \sqrt{(p_x^2 + \delta)^2 + p_y^2}$$

linear and quadratic dispersions in orthogonal directions

$\delta > 0$



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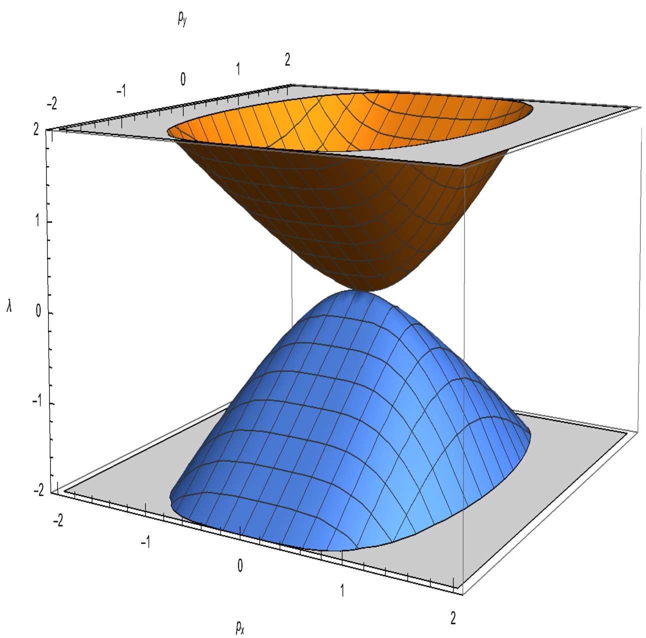
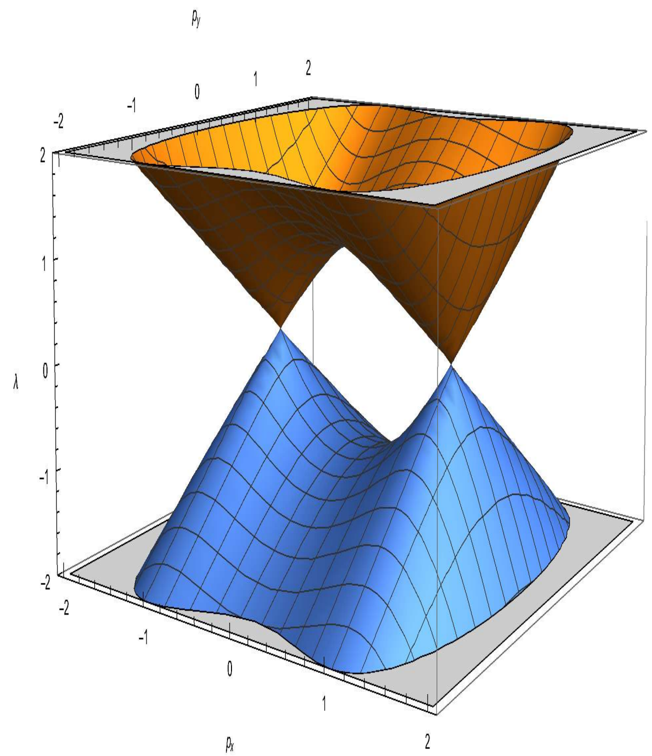


$\delta < 0$



[Pardo, Pickett 2009], [Banerjee, Singh, Pardo, Pickett 2009], [Delplace, Montambaux 2010]

Basic spectral properties

$$H_0 := \begin{pmatrix} -i\partial_y & -\partial_x^2 + \delta \\ -\partial_x^2 + \delta & i\partial_y \end{pmatrix}$$ in $\mathcal{H} := L^2(\mathbb{R}^2; \mathbb{C}^2)$

dom $H_0 := \left\{\psi \in \mathcal{H} : \ \partial_x \psi,\, \partial_x^2 \psi,\, \partial_y \psi \in \mathcal{H}\right\}$

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electromagnetic perturbations

$$H_\varepsilon := H_0 + \varepsilon V$$ $\varepsilon \geq 0$, $V := \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \in L^\infty(\mathbb{R}^2; \mathbb{C}^{2\times 2})$

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$\sigma_{\mathrm{disc}}(H_\varepsilon) \neq \varnothing$?

- $\lambda \in \sigma(H_\varepsilon) \quad \Longleftrightarrow \quad \lambda^2 \in \sigma(H_\varepsilon^2)$

Sufficient conditions

$$I_\varepsilon^+ := \int_{\mathbb{R}^2} (\varepsilon^2 |V_{11}|^2 + \varepsilon^2 |V_{12}|^2 + 2\delta\varepsilon\, \Re V_{12})$$

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Corollary. Let $\delta > 0$ and $\int_{\mathbb{R}^2} \Re V_{12} < 0$.

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Remark. The method additionally yields quantitative bounds.

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Multiplying both sides by $\overline{\psi}_1$, integrating over $\mathbb{R}^2$ and taking the real part yield:

$$\|\partial_x \psi_1\|^2 + \delta\,\|\psi_1\|^2 + (\psi_1, \varepsilon V_{21}\psi_1) + \|R^{1/2}\partial_y \psi_1\|^2 = E^2\,\|R^{1/2}\psi_1\|^2\,.$$

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Using $\|R^{1/2}\psi_1\|^2 \leq \|R\|\|\psi_1\|^2 = \delta^{-1}\|\psi_1\|^2$, one gets $\delta^2 \leq E^2$. *q.e.d.*

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Let $\{\varphi_j\}_{j=1}^{\infty}$ be *any* ONB of $L^2(\mathbb{R}^2)$ and write:

$$\psi_1 = \sum_{j=1}^{\infty} a_j \, \varphi_j \,, \qquad \psi_2 = \sum_{j=1}^{\infty} b_j \, \varphi_j \,.$$

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Then $H_\varepsilon \psi = E\psi$ in $\mathcal{H}$ $\iff$ $\begin{pmatrix} \boldsymbol{C}_{11} & \boldsymbol{C}_{12} \\ \boldsymbol{C}_{21} & \boldsymbol{C}_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} = E \begin{pmatrix} \boldsymbol{D} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{D} \end{pmatrix} \begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix}$ in ℓ^2.

$$\begin{aligned}
\boldsymbol{a} &:= \{a_j\}_{j=1}^{\infty} \\
\boldsymbol{b} &:= \{b_j\}_{j=1}^{\infty} \\
\boldsymbol{C}_{11} &:= \Big\{(\varphi_k, -i\partial_y \varphi_j) + (\varphi_k, \varepsilon V_{11} \varphi_j)\Big\}_{k,j=1}^{\infty} \\
\boldsymbol{C}_{12} &:= \Big\{(\varphi_k, (-\partial_x^2 + \delta)\varphi_j) + (\varphi_k, \varepsilon V_{12} \varphi_j)\Big\}_{k,j=1}^{\infty} \\
\boldsymbol{C}_{21} &:= \Big\{(\varphi_k, (-\partial_x^2 + \delta)\varphi_j) + (\varphi_k, \varepsilon V_{21} \varphi_j)\Big\}_{k,j=1}^{\infty} \\
\boldsymbol{C}_{22} &:= \Big\{(\varphi_k, i\partial_y \varphi_j) + (\varphi_k, \varepsilon V_{22} \varphi_j)\Big\}_{k,j=1}^{\infty} \\
\boldsymbol{D} &:= \Big\{(\varphi_k, \varphi_j)\Big\}_{k,j=1}^{\infty}
\end{aligned}$$

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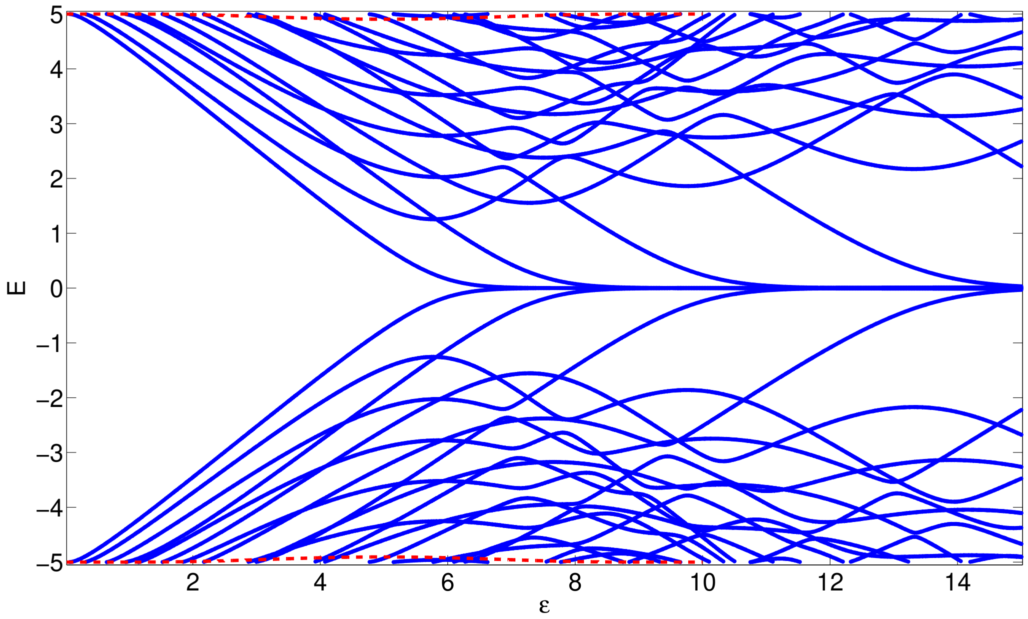
\boldsymbol{a} &:= \{a_j\}_{j=1}^{\infty} \\
\boldsymbol{b} &:= \{b_j\}_{j=1}^{\infty} \\
\boldsymbol{C_{11}} &:= \left\{(\varphi_k, -i\partial_y \varphi_j) + (\varphi_k, \varepsilon V_{11} \varphi_j)\right\}_{k,j=1}^{\infty} \\
\boldsymbol{C_{12}} &:= \left\{(\varphi_k, (-\partial_x^2 + \delta)\varphi_j) + (\varphi_k, \varepsilon V_{12} \varphi_j)\right\}_{k,j=1}^{\infty} \\
\boldsymbol{C_{21}} &:= \left\{(\varphi_k, (-\partial_x^2 + \delta)\varphi_j) + (\varphi_k, \varepsilon V_{21} \varphi_j)\right\}_{k,j=1}^{\infty} \\
\boldsymbol{C_{22}} &:= \left\{(\varphi_k, i\partial_y \varphi_j) + (\varphi_k, \varepsilon V_{22} \varphi_j)\right\}_{k,j=1}^{\infty} \\
\boldsymbol{D} &:= \left\{(\varphi_k, \varphi_j)\right\}_{k,j=1}^{\infty}
\end{aligned}$$

Numerical approximation: $\infty \longrightarrow N < \infty$.

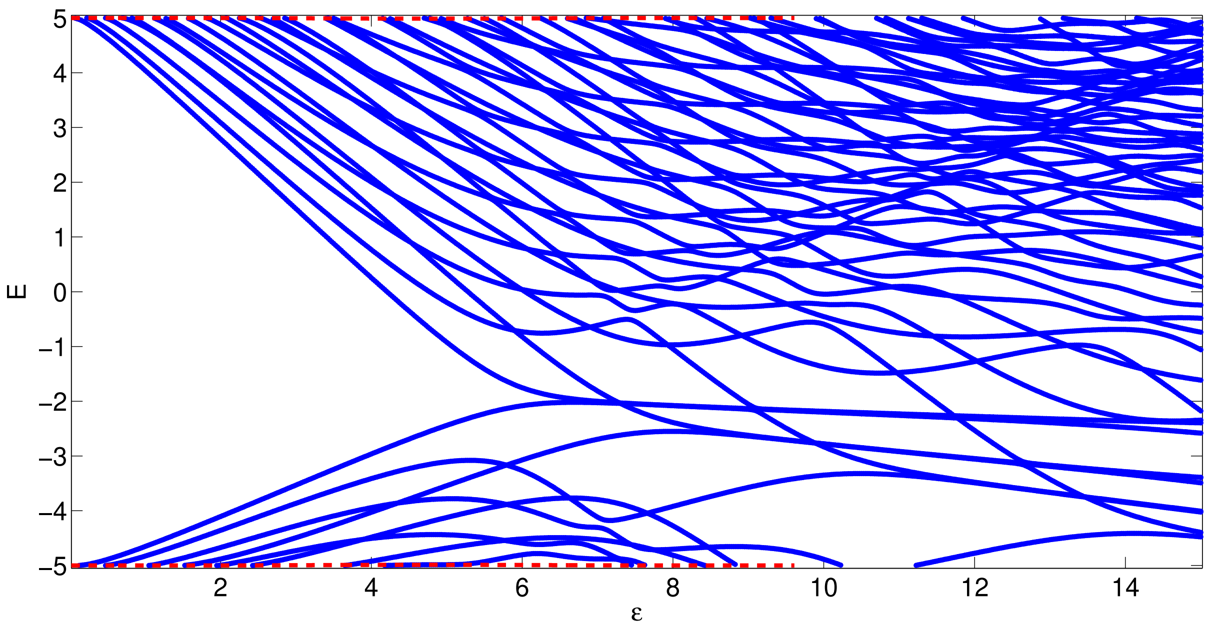
Numerical results: eigenvalues

$\delta = 5$



$V_{11} = 0 = V_{22}$

$V_{21} = -\chi_{B_2}$



$V_{21} = -\chi_{B_2}$

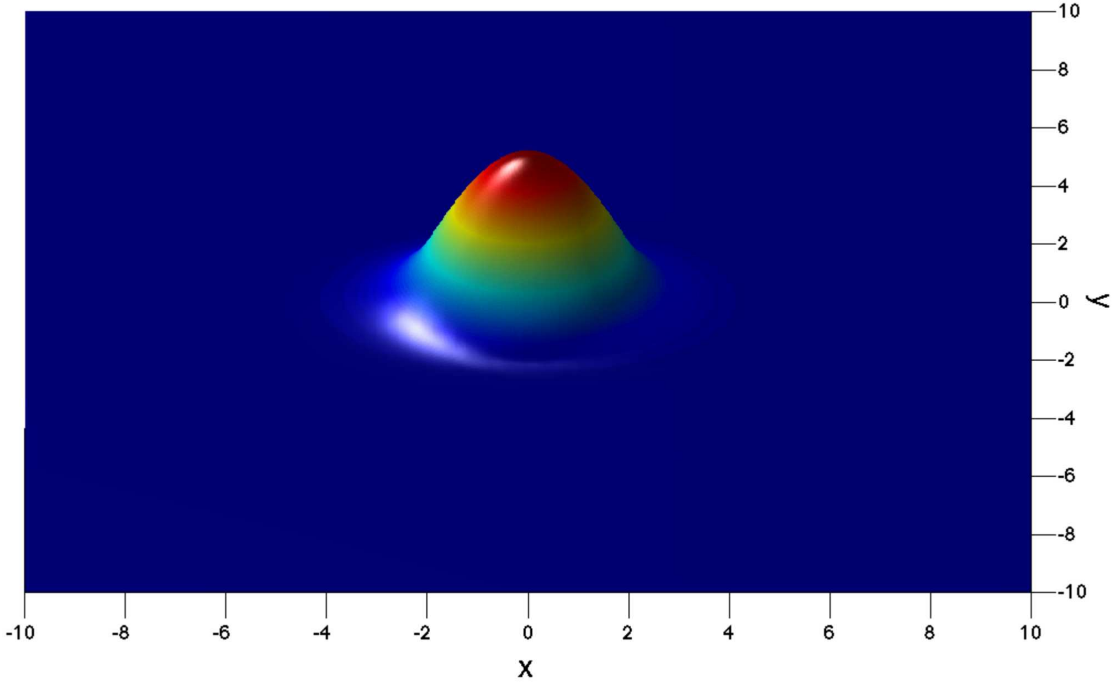
$V_{11} = 0.2\chi_{B_2}$

$V_{22} = -0.9\chi_{B_2}$

Numerical results: eigenfunctions

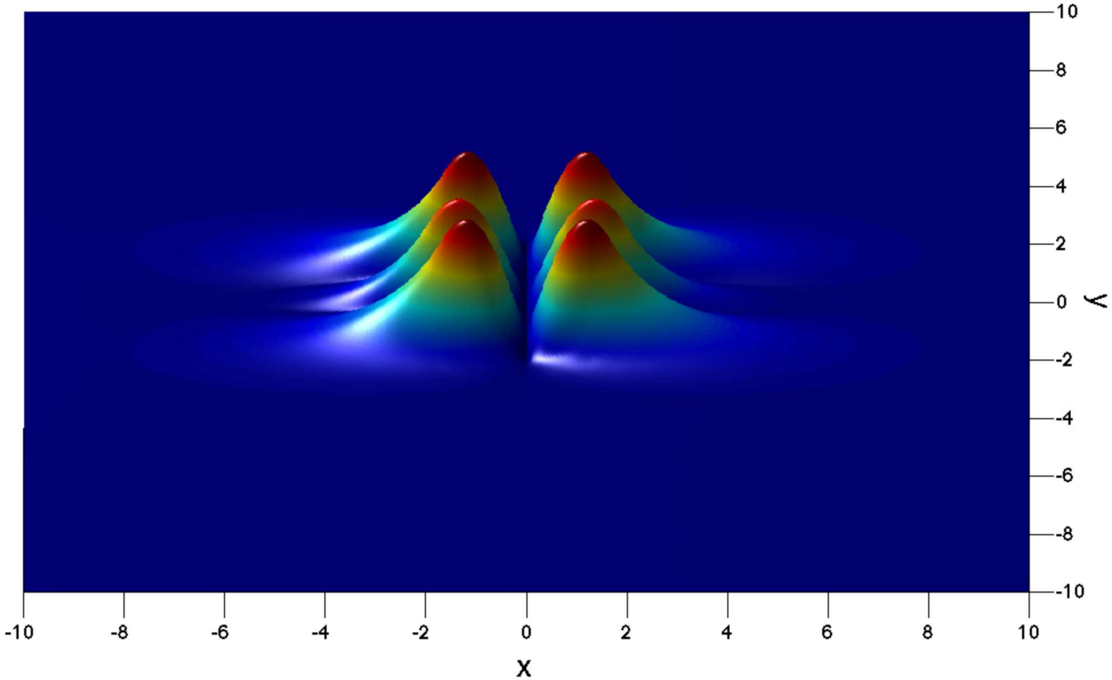
$\delta = 5$

$|\psi|$



$E \approx 2.9893$

$$V_{11} = 0 = V_{22}$$
$$V_{21} = -\chi_{B_2}$$
$$\varepsilon = 2.5$$



$E \approx 4.8284$

Conclusions

Semi-metal pearl square exquisite retro sunglasses (2021)

Conclusions

Moral : new type of hybrid materials:

simultaneously quadratic and linear dispersions

semi-metal *semi-Dirac*

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¿ waveguides, non-self-adjointness, *etc.* ?

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