Szegő-type inequality for the 2-D Dirac operator with infinite mass boundary conditions

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joint work with P. Antunes, R. Benguria, and T. Ourmières-Bonafos

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Outline

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Cauchy-Riemann differential expressions

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$$\mathrm{D}_\Omega u := \mathcal{D}u, \quad \mathrm{dom}\,\mathrm{D}_\Omega := \left\{ u \in H^1(\Omega, \mathbb{C}^2) \colon u_2|_{\partial\Omega} = \mathrm{i}(\nu_1 + \mathrm{i}\nu_2) u_1|_{\partial\Omega} \right\}.$$

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Principal eigenvalue

$E_1(\Omega) := \inf(\sigma(\mathsf{D}_\Omega) \cap \mathbb{R}_+)$ describes the size of the spectral gap.

We are interested in relation between $E_1(\Omega)$ and the shape of Ω.

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Faber-Krahn inequality among planar domains?

$$E_1(\Omega) \geq \sqrt{\frac{\pi}{|\Omega|}} E_1(\mathbb{D}), \qquad \text{(equality } \textit{iff } \Omega \text{ is a disk)}$$

In other words, **among all planar domains of fixed area the disk minimizes the principal eigenvalue of D_Ω.**

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with equality if and only if Ω *is a disk.*

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Theorem (Stronger version, ABLO-20)

$$E_1(\Omega) \leq \frac{|\partial\Omega| + \sqrt{|\partial\Omega|^2 + 8\pi E_1(\mathbb{D})(E_1(\mathbb{D}) - 1)(\pi r_{\rm i}^2 + |\Omega|)}}{2(\pi r_{\rm i}^2 + |\Omega|)}$$

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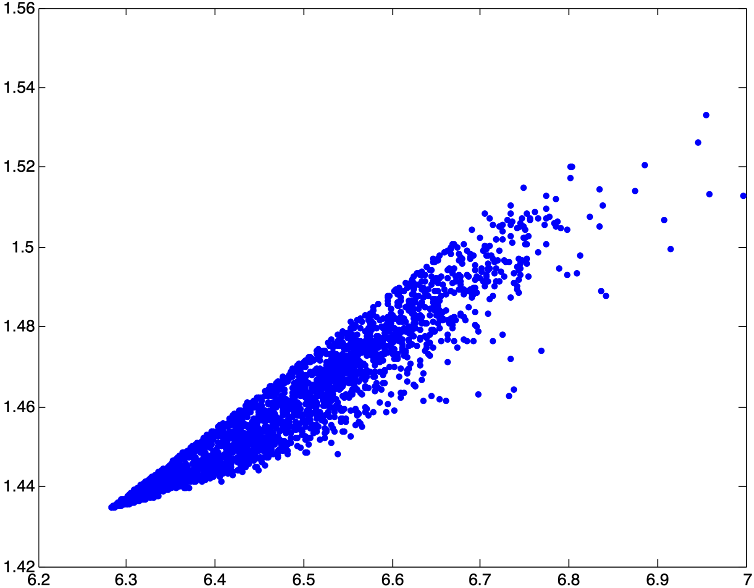

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Assume that these identities are true up to $\partial\Omega$ & use that $v = \mathrm{i}\mathbf{n}u$ on $\partial\Omega$:

$$\begin{cases} -4\partial_z\partial_{\bar z} u = E^2 u, & \text{in } \Omega \\ \overline{\mathbf{n}}\partial_{\bar z} u + \frac{E}{2} u = 0, & \text{on } \partial\Omega \end{cases} \qquad (\star)$$

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The spectral problem $(\star)$ is equivalent

$$\begin{cases} -\Delta u = E^2 u & \text{in } \Omega, \\ \partial_n u + \mathrm{i}\partial_t u + Eu = 0 & \text{on } \partial\Omega, \end{cases}$$

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Quadratic form

$$\mathfrak{q}^{\Omega}_{E,0}[u] := 4\int_{\Omega} |\partial_{\bar{z}} u|^2 \mathrm{d}x - E^2 \int_{\Omega} |u|^2 \mathrm{d}x + E \int_{\partial\Omega} |u|^2 \mathrm{d}s,$$

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Theorem (Non-linear variational characterization of $E_1(\Omega)$)

$E > 0$ is the first non-negative eigenvalue of D_{Ω} if and only if $\mu^{\Omega}(E) = 0$.

Properties of $\mu^\Omega(E)$

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Proposition

1. μ^Ω *is a continuous and concave function on* $\mathbb{R}_+$.
2. $\mu^\Omega(0) = 0$ *and there exists* $E_\star^\Omega > 0$ *s.t.* $\mu^\Omega(E) > 0$ *for all* $E \in (0, E_\star^\Omega)$.

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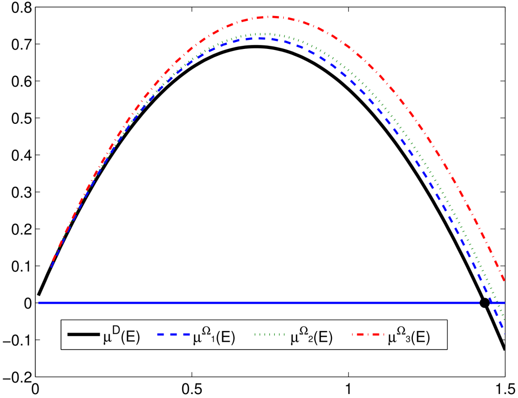

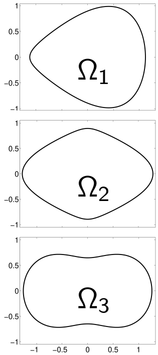

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For $x \in \mathbb{D}$, consider $u_0(x) = J_0\big(E_1(\mathbb{D})|x|\big) \in H^1(\mathbb{D}) \subset \mathrm{dom}\, \mathfrak{q}^{\mathbb{D}}_{E_1(\mathbb{D})}$.

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For $x = (x_1, x_2) \in \Omega$, $v_0(x_1, x_2) = u_0(f^{-1}(x_1 + \mathrm{i}x_2)) \in H^1(\Omega) \subset \mathrm{dom}\,\mathfrak{q}^{\Omega}_{E}$

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By the min-max principle, there holds

$$\mu^\Omega(E) \le \frac{\mathfrak{q}_E^\Omega[v_0]}{\|v_0\|^2_{L^2(\Omega)}} = \frac{\|\nabla v_0\|^2_{L^2(\Omega)} + E\|v_0\|^2_{L^2(\partial\Omega)}}{\|v_0\|^2_{L^2(\Omega)}} - E^2,$$

where we used that v_0 is real-valued to get $\|\nabla v_0\|^2_{L^2(\Omega)} = 4\|\partial_{\bar z} v_0\|^2_{L^2(\Omega)}$.

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$$\begin{aligned}\|\nabla v_0\|^2_{L^2(\Omega)} &= \|\nabla u_0\|^2_{L^2(\mathbb{D})} = 2\pi E_1(\mathbb{D})^2 \int_0^1 J_1(E_1(\mathbb{D})r)^2 r\,\mathrm{d}r \\ &= 2\pi J_0\big(E_1(\mathbb{D})\big)^2 E_1(\mathbb{D})\big(E_1(\mathbb{D}) - 1\big) \\ \|v_0\|^2_{L^2(\partial\Omega)} &= |\partial\Omega| J_0(E_1(\mathbb{D}))^2 \\ \|v_0\|^2_{L^2(\Omega)} &\ge J_0\big(E_1(\mathbb{D})\big)^2(\pi r_{\rm i}^2 + |\Omega|)\end{aligned}$$

Estimate of $\mu^\Omega(E)$ II

Polynomial

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$$\mu^\Omega(E) \le \frac{P(E)}{\pi r_{\rm i}^2 + |\Omega|} \qquad \text{(final estimate)}$$

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$$E_{\mathrm{crit}} := \frac{|\partial\Omega| + \sqrt{|\partial\Omega|^2 + 8\pi E_1(\mathbb{D})\big(E_1(\mathbb{D}) - 1\big)(\pi r_{\mathrm{i}}^2 + |\Omega|)}}{2(\pi r_{\mathrm{i}}^2 + |\Omega|)}.$$

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Hence, we conclude from the final estimate that

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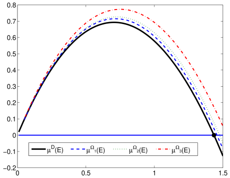
Remembering the properties of $\mu^\Omega(E)$ we end up with

$$E_1(\Omega) \le E_{\rm crit}$$

by which the proof is complete.



Thank you

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P. Antunes, R. Benguria, V. L., T. Ourmières-Bonafos, A variational formulation for Dirac operators in bounded domains. Applications to spectral geometric inequalities, to appear in Comm. Math. Phys..

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