Diverging eigenvalues in domain truncations of Schrödinger operators with complex potentials

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joint work with Petr Siegl [3]

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Outline

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Domain truncation for $T = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mathrm{i}x^3$ on $L^2(\mathbb{R})$

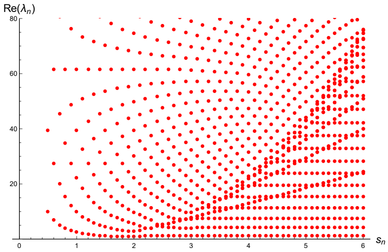


Figure: Real part of the spectrum of T_n truncated to $L^2((-s_n, s_n))$

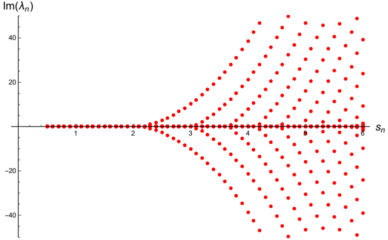


Figure: Imaginary part of the spectrum of T_n truncated to $L^2((-s_n, s_n))$

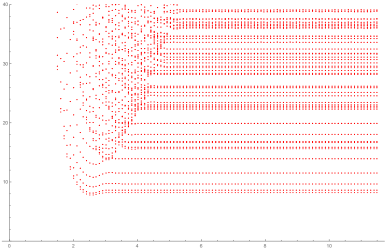
- Bender, Boetcher 1998 [3]: introduction of the problem, real spectra

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (\mathrm{i}x)^N, \quad N \geq 2$$

- Boegli, Siegl, Tretter 2017 [4] - domain truncations is spectrally exact
- Günther, Stefani 2019 [9]: description of diverging eigenvalues for $N = 2k - 1, k \in \mathbb{N}$ (analytic WKB and Stokes graph analysis).

Brown, Marletta 2004 [5]: truncation of 2D radial potential

$$T = -\Delta_D + (1+3\mathrm{i})(x^2+y^2), \quad \text{on } L^2(\mathbb{R}^2 \setminus \overline{B_1})$$
$$T_n = -\Delta_D + (1+3\mathrm{i})(x^2+y^2), \quad \text{on } L^2(B_{s_n} \setminus \overline{B_1})$$



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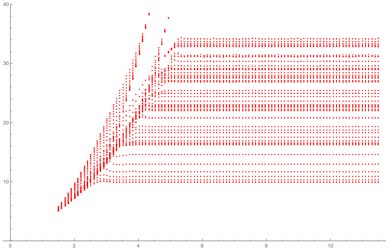

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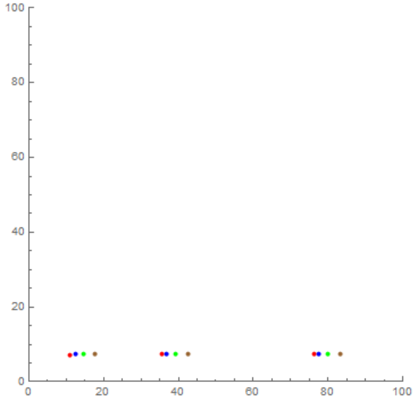
Decomposed to 1D problems

$$T_{n,l} = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + (1+3\mathrm{i})r^2 + \frac{l^2}{r^2}, \quad r \in [1, s_n], \quad l \in \mathbb{N}$$



Enhanced dissipation: $T_g = -\frac{d^2}{dx^2} + x^2 + igf(x)$ on $L^2(\mathbb{R})$

- fluid mechanics, behaviour of $\inf \mathrm{Re}(\lambda)$
- Gallagher, Gallay, Nier 2009 [8], Schenker 2011 [10]:
  $f(x) = \frac{1}{1+|x|^\kappa}, \quad \kappa > 0$

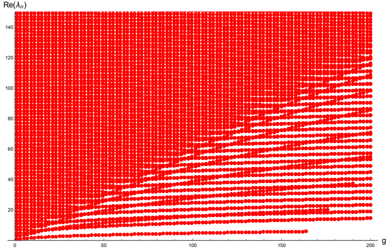
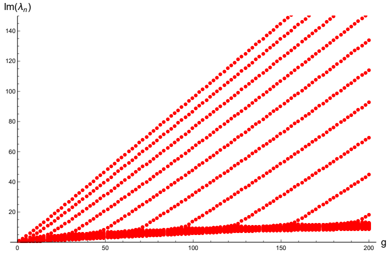


Figure: Real part of the spectrum of T_g for $f(x) = \frac{1}{1+|x|^\kappa}$ with $\kappa = 3.15$



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$\mathcal{PT}$-symmetric phase transition

- Baker, Mityagin (2020) [1]:

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2 + \mathrm{i}g(\delta(x-b) - \delta(x+b))$$

- Caliceti, Graffi (2014) [6]:

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{x^{2M}}{2M} - \tilde{g}\frac{x^{M-1}}{M-1}, \quad \tilde{g} \in \mathbb{C}, \quad M = 2, 4, 6, \ldots$$

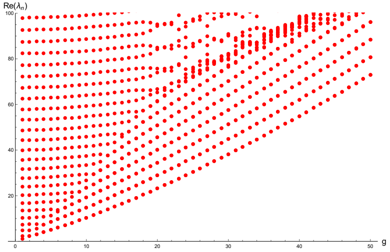
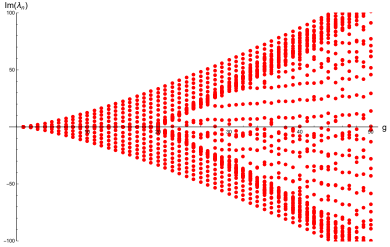


Figure: Real part of the spectrum for $\tilde{g} = \mathrm{i}g$, $M = 2$



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How to describe such diverging eigenvalues?

Example of transformations: $T = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mathrm{i}x^3$, acting in $L^2(\mathbb{R})$

- Domain truncation (spectrally exact approximation of T)

$$T_n = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mathrm{i}x^3 , \quad \mathrm{Dom}(T_n) = \{f \in W^{2,2}(-n,n) : f(\pm n) = 0\}$$

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- Translation[1] $x \to x - n$, unitary op. $(\mathcal{N}f)(x) = f(x-n)$

$$\mathcal{N} T_n \mathcal{N}^{-1} = \tilde{T}_n = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mathrm{i}(x-n)^3 , \quad \text{acting in } L^2((0,2n))$$

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- Scaling $x \to n^\alpha y$ unitary op.,
  $\mathcal{R} : L^2((0,2n)) \to L^2((0,2n^{1+\alpha})) : f(x) \mapsto n^{-\alpha/2} f(xn^{-\alpha})$

$$\mathcal{R}\tilde{T}_n\mathcal{R}^{-1} = n^{2\alpha}\left[-\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \mathrm{i}(n^{5\alpha}y^3 - 3n^{4\alpha+1}y^2 + 3n^{3\alpha+2}y - n^{2\alpha+3})\right]$$

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- Choice of $\alpha = -2/3$

$$\tilde{S}_n = n^{4/3}\left[S_n - \mathrm{i}n^{5/3}\right]\ , \quad \text{acting in } L^2((0,2n^{1/3}))$$

where

$$S_n = -\frac{\mathrm{d}^2}{\mathrm{d}y^2} + 3\mathrm{i}y + \mathrm{i}n^{-10/3}y^3 - 3\mathrm{i}n^{-5/3}y^2$$

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- S_n converges to the Airy operator on half-line

$$S = -\frac{\mathrm{d}^2}{\mathrm{d}y^2} + 3\mathrm{i}y \ , \quad \mathrm{Dom}(S) = \{f \in W^{2,2}(\mathbb{R}^+) : xf \in L^2(\mathbb{R}^+), f(0) = 0\}$$

$$\sigma(S) = \{\nu_k\}_{k\in\mathbb{N}} \implies \nu_k + r_{k,n} \in \sigma(S_n), \forall n > n_k$$

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- EVS of T_n: as $T_n = \mathcal{N}^{-1}\mathcal{R}^{-1}[n^{4/3}(S_n - \mathrm{i}n^{5/3})]\mathcal{N}\mathcal{R}$

$$\lambda_{k,n} = n^{4/3}(\underbrace{\nu_k + r_{k,n}}_{\in\sigma(S_n)} - \mathrm{i}n^{5/3}), \quad \text{as } n \to \infty$$

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convergence of isolated eigenvalues of $S_n \to S$?

Outline

Assumption

Suppose that

1. domains $\{\Omega_n\}_{n\in\mathbb{N}^*} \subset \mathbb{R}^d$ are open and non-empty, Ω_∞ is unbounded and (not necessarily bounded) domains $\{\Omega_n\} \subset \Omega_\infty$ exhaust Ω_∞ as $n \to \infty$;

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2. potentials $Q_n \in W^{1,\infty}_{\mathrm{loc}}(\Omega_n)$ with $\operatorname{Re} Q_n \geq 0$, $n \in \mathbb{N}^*$, satisfy uniformly

$$\exists \varepsilon_\nabla \in [0, \epsilon_{crit}),\ \exists M_\nabla \geq 0,\ \forall n \in \mathbb{N}^*,\ |\nabla Q_n| \leq \varepsilon_\nabla |Q_n|^{\frac{3}{2}} + M_\nabla \quad \text{a.e. in } \Omega_n, \tag{1}$$

For self-adjoint case $\epsilon_{crit} = 2$ (Everitt, Giertz 1978 [7]) . For non-self-adjoint case $\epsilon_{crit} \in [2-\sqrt{2}, 2]$, where $2-\sqrt{2} \approx 0.5857$;

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3. operators $T_n = -\Delta + Q_n$ in $L^2(\Omega_n)$ are introduced via quadratic forms t_n, $n \in \mathbb{N}^*$ and cut-offs $\{\xi_n\}_{n\in\mathbb{N}}$, $\xi_n : \Omega_\infty \to [0,1]$, $\chi_{\Omega_n}(x)\xi_n(x) = \xi_n(x)$ are such that

$$\sup_{n\in\mathbb{N}} \left(\| |\nabla \xi_n| \|_{L^\infty} + \|\Delta \xi_n\|_{L^\infty} \right) < \infty. \tag{2}$$

   Furthermore

$$\begin{aligned} &\forall f_n \in \operatorname{Dom}(T_n), \quad \xi_n f_n \in \operatorname{Dom}(t_\infty), \\ &\forall g \in \operatorname{Dom}(T_\infty), \quad \xi_n g \in \operatorname{Dom}(t_n); \end{aligned} \tag{3}$$

4. potentials $\{Q_n\}$ converge in the following sense

$$\tau_n := \left\| \frac{\xi_n(Q_n - Q_\infty)}{(Q_n+1)(Q_\infty+1)} \right\|_{L^\infty(\Omega_n)} + \left\| \frac{\zeta_n}{Q_n+1} \right\|_{L^\infty(\Omega_n)} + \left\| \frac{\zeta_n}{Q_\infty+1} \right\|_{L^\infty(\Omega_\infty)} = o(1), \quad n \to \infty, \tag{4}$$

where

$$\zeta_n := \chi_{\operatorname{supp} \tilde{\xi}_n}, \qquad \tilde{\xi}_n := 1 - \xi_n, \quad n \in \mathbb{N}. \tag{5}$$

■

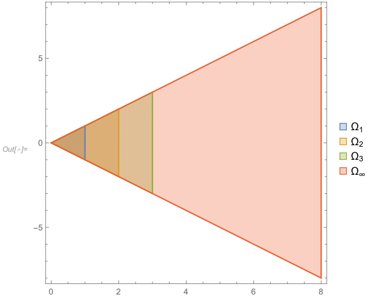
Example

Sequence of potentials

$$Q_n(x,y) = \mathrm{i}(x^2 + y^2) + W_n(x,y), \quad ||W_n||_{L^\infty} \to 0 \text{ as } n \to \infty$$

and domains

$$\Omega_n = \{(x,y) : y < x, -y < x, x \in (0,n)\}$$



Theorem

1. $\{T_n\}$ converge to T_∞ in the generalized norm resolvent sense (and hence there is no spectral pollution): for every $z \in \rho(T_\infty)$, there is $n_z > 0$ such that $z \in \rho(T_n)$, $n > n_z$ and

$$\|(T_n - z)^{-1}\chi_{\Omega_n} - (T_\infty - z)^{-1}\|_{\mathcal{B}(L^2(\Omega_\infty))} = \mathcal{O}_z(\tau_n), \quad n \to \infty; \qquad (6)$$

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2. spectral projections converge in norm:

$$\|E_{k,n} - E_k\|_{\mathcal{B}(L^2(\Omega_\infty))} = \mathcal{O}_k(\tau_n), \quad n \to \infty; \tag{7}$$

where

$$E_k := \frac{1}{2\pi \mathrm{i}} \int_{\gamma_k} (z - T_\infty)^{-1} \, \mathrm{d}z, \qquad E_{k,n} := \frac{1}{2\pi \mathrm{i}} \int_{\gamma_k} (z - T_n)^{-1} \chi_{\Omega_n} \, \mathrm{d}z,$$

Theorem

3. spectral inclusion for isolated eigenvalues: for every $\nu_k \in \sigma_{\mathrm{disc}}(T_\infty)$, as $n \to \infty$, there are eigenvalues $\nu_{k,n}$ of T_n in a neighborhood of ν_k and (for simple eigenvalues)

$$|\nu_k - \nu_{k,n}| = \mathcal{O}_k\left(\kappa_n\right), \quad n \to \infty, \tag{8}$$

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4. (generalized) eigenvectors converge in norm: for every $\psi \in \mathrm{Ran}(E_k)$ as $n \to \infty$

$$\|\psi - E_{k,n}\psi\| = \mathcal{O}_k\left(\kappa_n\right), \quad n \to \infty. \tag{9}$$

where

$$\kappa_n := \max_{\substack{\phi \in \mathrm{Ran} E_k \\ \|\phi\|=1}} \left(\left\| \frac{\xi_n(Q_n - Q_\infty)}{(Q_n+1)(Q_\infty+1)}\phi \right\| + \|\zeta_n \phi\| \right). \tag{10}$$

Application of the theorem to the domain truncation

Boegli, Siegl, Tretter 2017 [4]:
proved spectral exactness of domain truncation technique on $\mathbb{R}^d$ and exterior domains for wide classes of complex potentials, of approximating domains Ω_n, and of boundary conditions on $\partial\Omega_n$ such as mixed Dirichlet/Robin type.

New results:

- perturbed potential/sequence of potentials
- broader class of unbounded limit domains Ω_∞ (e.g. cone)
- Ω_n can be unbounded, T_∞ with non-compact resolvent (e.g. $\Omega_n = (-\infty, n)$, $Q(x) = \mathrm{i}e^x$)
- rate of convergence for resolvents

Outline

$T = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mathrm{i}x^3$ on $L^2(\mathbb{R})$

Eigenvalues lying asymptotically in the spectra of truncated operators T_n

$$\lambda_{k,n}^{(1)} = (3s_n)^{2/3}(\nu_k + \mathcal{O}_k(s_n^{-5/3})) - \mathrm{i}s_n^3, \quad \lambda_{k,n}^{(2)} = \overline{\lambda_{k,n}^{(1)}}$$

where $\nu_k = e^{-2\pi\mathrm{i}/3}\mu_k$, $\mathrm{Ai}(\mu_k) = 0$

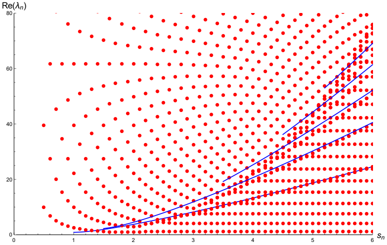


Figure: Real part of spectrum, real part of first 5 asymptotic curves $\lambda_{k,n}^{(1)}, \lambda_{k,n}^{(2)}$.

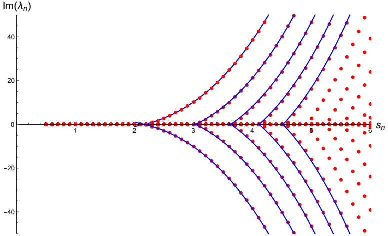
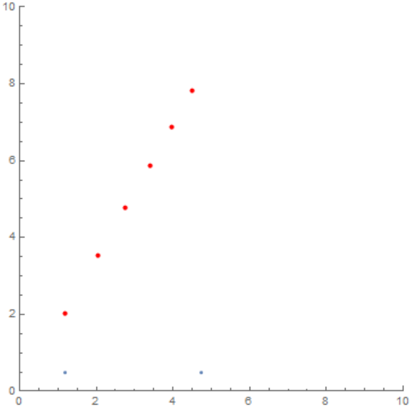


Figure: Imaginary part of spectrum, imaginary part of first 5 asymptotic curves $\lambda_{k,n}^{(1)}, \lambda_{k,n}^{(2)}$.

Rate of the convergence

In complex plane are shown: red dots ν_k and blue dots $(\lambda_{k,n} + is_n^3)3^{-\frac{2}{3}} s_n^{-\frac{4}{3}}$



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Eigenvalues lying asymptotically in the spectra of truncated operators T_n

$$\lambda_{k,n} = e^{2s_n/3}\overline{(\nu_k + \mathcal{O}_k(\mathrm{e}^{-s_n/3}))} + \mathrm{i}\mathrm{e}^{s_n}$$

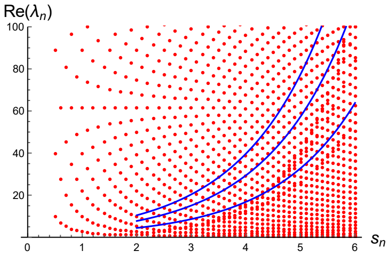


Figure: Real part of spectrum

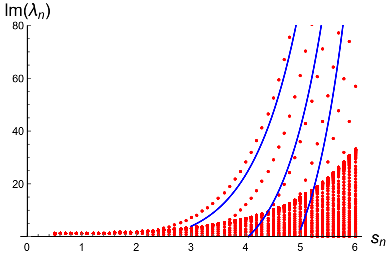


Figure: Imaginary part of spectrum

Radially symmetric potential on annuli - Brown, Marletta (2004)

$$T = -\Delta_D + \mathrm{i}|x|^2, \quad \text{on } L^2(\mathbb{R}^d \setminus \overline{B_1(0)})$$

truncated and decomposed to 1D problems

$$T_{n,l} = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \mathrm{i}r^2 + \frac{(d-1)(d-3)/4 + l(l+d-2)}{r^2}, \quad \text{in } L^2((1, s_n))$$

$$\lambda_{k,n,l} = (2s_n)^{\frac{2}{3}} \left(\overline{\nu_k} + \mathcal{O}_{k,l}\left(s_n^{-\frac{4}{3}} \right) \right) + \mathrm{i}s_n^2, \quad n \to \infty;$$

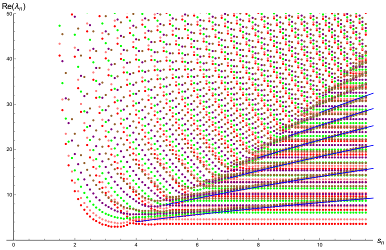
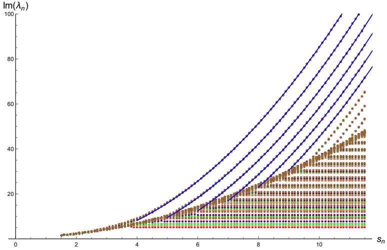


Figure: Real part of spectrum of $T_{n,l}$ for $d = 3$ and $l = 1, 2, 3, 4, 5$



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Higher dimensions

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- 2D rotated squares and polynomial potential

$$Q(x,y) = \mathrm{i}(x^3 + y^4) + x^2y^2, \quad x,y \in \mathbb{R}$$

and a sequence of $\pi/4$ rotated squares Ω_n. Then spectra of T_n contain asymptotically the eigenvalues

$$\lambda_{k,n}^{(j)} = (3s_n^2)^{\frac{2}{3}} \left(\nu_k + \rho_{k,n}^{(j)}\right) - \mathrm{i}s_n^3, \quad n \to \infty,$$

where $\{\nu_k\}$ are eigenvalues of the complex Airy operator in a sector

$$S_A := -\Delta + \mathrm{i}x, \quad \mathrm{Dom}(S_A) := \mathrm{Dom}(\Delta_{\mathrm{D}}) \cap \mathrm{Dom}(|x|),$$

$T_g = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2 + \mathrm{i}g\frac{1}{1+|x|^\kappa}$ on $L^2(\mathbb{R})$

Schenker [10]:

$$\mathrm{Re}(\lambda) \geq C|g|^{2/(2+\kappa)}$$

Our result (optimality)

$$\lambda_{k,g} = g^{\frac{2}{2+\kappa}}(\nu_k^\kappa + r_{k,g}^\kappa) + \mathrm{i}g$$

where $\nu_k \in \sigma(T\kappa)$, $T_\kappa = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \mathrm{i}|x|^\kappa$

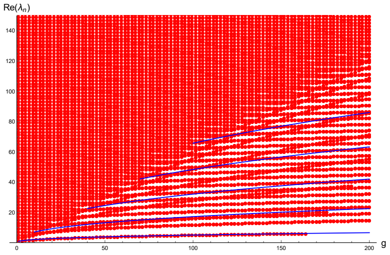


Figure: Real part of spectrum of T_g for $\kappa = 3.15$

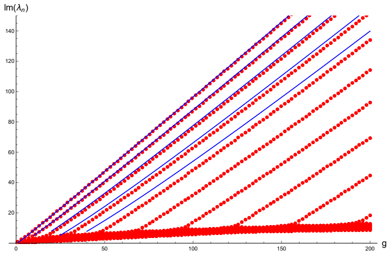


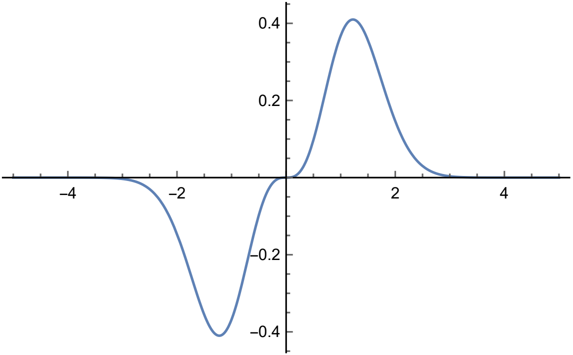
Figure: Imaginary part of spectrum of T_g for $\kappa = 3.15$

Baker, Mityagin (2020): $T_g = -\frac{d^2}{dx^2} + x^2 + ig(\delta(x-b) - \delta(x+b))$

- proved that the number of non-real eigenvalues diverges as $g \to \infty$
- "smooth" version of BM potential

$$T_g = -\frac{d^2}{dx^2} + x^2 + igx^3 e^{-x^2} \text{ in } L^2(\mathbb{R}) \tag{11}$$

- stationary points $x_0 = 0$, $x_1 = -\sqrt{\frac{3}{2}}$ and $x_2 = -x_1$



- stationary point x_0

$$\lambda_{k,g}^{(x_0)} = g^{\frac{2}{5}}(\nu_k + \mathcal{O}_k(g^{-\frac{6}{25}})), \quad g \to +\infty, \tag{12}$$

where ν_k are eigenvalues of imaginary cubic oscillator.

- stationary points x_1, x_2

$$\lambda_{k,g}^{(x_1)} = g^{\frac{1}{2}}(\nu_k + \mathcal{O}_k(g^{-\frac{1}{8}})) - \mathrm{i}g(2\mathrm{e})^{-\frac{1}{2}} + \frac{3}{2}, \quad \lambda_{k,g}^{(x_2)} = \overline{\lambda_{k,g}^{(x_1)}} \quad g \to +\infty, \tag{13}$$

where $\nu_k = (\frac{27}{2\mathrm{e}^3})^{\frac{1}{4}} \mathrm{e}^{\mathrm{i}\frac{\pi}{4}}(2k+1), \ k \in \mathbb{N}_0$,

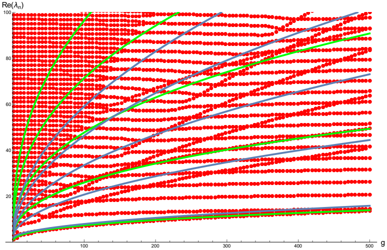


Figure: Real part of the spectrum of T_g

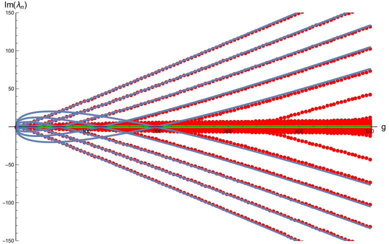


Figure: Imaginary part of the spectrum of T_g

Caliceti, Graffi (2014)

studied $\mathcal{PT}$-symmetric phase transitions for a class of operators in $L^2(\mathbb{R})$

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{x^{2M}}{2M} + \mathrm{i}g\,\frac{x^{M-1}}{M-1}, \quad M \in 2\mathbb{N} \tag{14}$$

rescale $x \mapsto g^{2M/(M+1)}x$ to obtain

$$\frac{1}{g^{\frac{2}{M+1}}}\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + g^2\left(\frac{x^{2M}}{2M} + \mathrm{i}\,\frac{x^{M-1}}{M-1}\right)\right] \tag{15}$$

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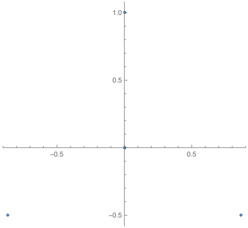
$M = 2$ case: stationary points $x_0 = 0$, $x_1 = \mathrm{i}$, $x_2 = \mathrm{e}^{\mathrm{i}\frac{7}{6}\pi}$ and $x_3 = \mathrm{e}^{\mathrm{i}\frac{11}{6}\pi}$



M=2

$$\lambda_{k,g}^{(x_3)} = \sqrt{\frac{3}{2}} g^{\frac{1}{3}} (\nu_k + \mathcal{O}_k(g^{-\frac{1}{6}})) + \frac{3}{4} \mathrm{e}^{\mathrm{i}\frac{5\pi}{3}} g^{\frac{4}{3}}, \quad \lambda_{k,g}^{(x_2)} = \overline{\lambda_{k,g}^{(x_3)}}, \quad g \to +\infty,$$

where $\nu_k = \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}(2k+1)$, $k \in \mathbb{N}_0$,

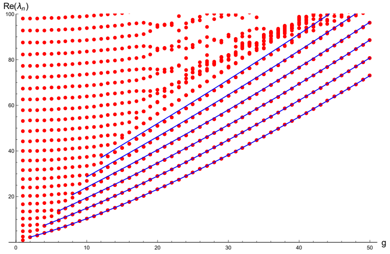


Figure: Real part of the spectrum

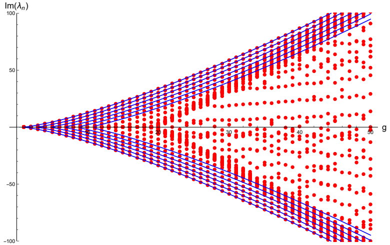


Figure: Imaginary part of the spectrum

$M \geq 4$

stationary point $x_0 = 0$ yields sequences of eigenvalues

$$\lambda^{(x_0)}_{k,g,M} = g^{-\frac{2}{M+1}}(\nu_{k,M} + \mathcal{O}(g^{-\frac{2}{M+1}})), \quad g \to +\infty, \tag{16}$$

multiple complex stationary points $x_k = \mathrm{e}^{\mathrm{i}\frac{4k-1}{2(M+1)}\pi}, \quad k = 1, \ldots, M+1$

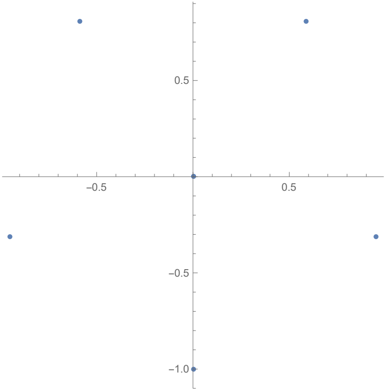


Figure: Stationary points for $M = 4$

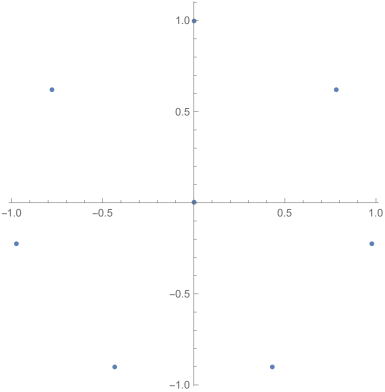


Figure: Stationary points for $M = 6$

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