# Painting a Picture of Surreal Numbers

## Kombinatorika na slovech a matematická fyzika 2021

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What follows is an illustration of surreal numbers, as given by John H. Conway. We shall adhere to:

- Conway's 1976 book **On Numbers And Games** (ONAG)
- Donald E. Knuth's 1974 book **Surreal Numbers: How Two Ex-Students Turned on to Pure Mathematics and Found Total Happiness** (SN:HTESTPMFTH)

*"In the beginning, everything was void, and J. H. W. H. Conway began to create numbers. Conway said, "Let there be two rules which bring forth all numbers large and small.""*
– SN:HTESTPMFTH

## Rule 1

Let $L, R$ be two sets of numbers such that no member of $L$ is greater or equal that any member of $R$. Then $x := \{L|R\}$ is a number. All numbers are created this way.

We also denote elements of $L$ as $x^L$ and elements of $R$ as $x^R$. We call $x^L$ a *left option* of $x$ and $x^R$ a *right option* of $x$.

## Rule 2

Let $x, y$ be two numbers. We say that $x \geq y$ iff no $x^R \leq y$ and $x \leq$ no $y^L$.

We also say that $x \leq y$ iff $y \geq x$. Furthemore we write

- $x = y$ iff $x \geq y$ & $x \leq y$
- $x < y$ iff $x \leq y$ & $x \not\geq y$.

It is important to stress that "$=$" is a defined relation and $x = y$ does not necessarily mean that $x$ and $y$ are *identical*, i. e. that their left and right options correspond exactly.

# Let's try to make some numbers

- Since we have no numbers to begin with, our only options for sets $L$ and $R$ are empty sets. But why not? We denote

$$\{\emptyset|\emptyset\} =: \{\,|\,\} =: 0.$$

  and we have our first number.

- Let us check whether $0 \geq 0$. That is true, unless some $0^R \leq 0$ or $0 \leq$ some $0^L$. But there are no $0^L$ or $0^R$...

- In the second step we can take $L, R$ as eighter empty sets or $\{0\}$. That gives us these 4 options

$$\{\,|\,\}, \quad \{0|\,\}, \quad \{\,|0\}, \quad \{0|0\}.$$

However, the first one is our 0, and the last one cannot be a number since $0 \geq 0$ (contradiction of Rule 1).
- We denote

$$\{0|\,\} =: 1,$$
$$\{\,|0\} =: -1.$$

And we could go on like this forever[1]. It can be shown (see ONAG, for example) that any number is strictly greater, resp. smaller than any of its left, resp. right options. Formally put:

### Theorem

For any number $x$ there holds $x^L < x < x^R$, for any $x^L$, $x^R$.

For example this means, that if we take $L$ to be any set of numbers, then $\{L|\}$ is a number strictly greater than any number in $L$.

[1] And beyond, as we shall see.

# Addition

Consider two numbers $x, y$. Their sum is defined as

$$x + y := \{x^L + y, x + y^L | x^R + y, x + y^R\}.$$

- for any number $x$ there holds $x + 0 \equiv \{x^L + 0 | x^R + 0\} = x$. In Conway's approach this is done inductively (much akin to the definition of numbers), i.e. one assumes this statement holds for all $x^L$ and $x^R$ and shows it then holds also for $x$.
- We find natural numbers amongst the surreals by starting with 1 and successively adding one.

## Multiplication

Consider two numbers $x, y$. We define their product as

$$xy = \{x^L y + xy^L - x^L y^L, x^R y + xy^R - x^R y^R \\ | x^L y + xy^R - x^L y^R, x^R y + xy^L - x^R y^L\}.$$

As we should expect, $x0 = 0$ and $x1 = x$ for all numbers $x$. As an excercise, let's multiply $\{0|1\}$ with $2 = \{1|\,\}$.

$$\{0|1\}\{1|\,\} = \{\{0|1\}\,|\,1 + \{0|1\}\}.$$

What kind of a number is that?

## Conway's Simplicity Theorem

Let $L$ and $R$ be two sets of numbers such that $L < R$. Then $x = \{L|R\}$ is the *earliest created* number satisfying $L < \{x\} < R$.

Using this theorem we find that our unknown number from the previous page is in fact the number 1, and therefore

$$\{0|1\} = \frac{1}{2}.$$

In fact, one can show (see ONAG) the following highly untrivial fact.

### The Field $\mathtt{No}$

The proper Class of all surreal numbers forms a totally ordered Field. This field is denoted as $\mathtt{No}$.

And since one has integers within the surreals, one therefore also has rational numbers. The Field $\mathtt{No}$, however, contains much more.

## Ordinal numbers

We say that $x$ is an ordinal number iff

$$x = \{L|\,\},$$

for some set of numbers $L$.

- Any $x \in \mathbb{N}_0$ is an ordinal.
- The first infinite ordinal is $\omega = \{1, 2, 3, \ldots \,|\,\}$.

- Return for a moment to where we constructed our first numbers. We have seen that first was created 0. We say that 0 was *born on day 0*. The numbers $1, -1$ were born on day one. The numbers $1/2$ and 2 (and their negatives) were born on day two.
- All days are labeled by ordinals – the greatest numbers created that day. The number $\omega$ was therefore born on day $\omega$.

# Real numbers

We say that the number $x$ is a real number iff $-n < x < n$ for some integer $n$ and

$$x = \{x - 1, x - \frac{1}{2}, x - \frac{1}{3}, \ldots \mid x + 1, x + \frac{1}{2}, x + \frac{1}{3}, \ldots\}$$

- Every rational number of the form $m/2^n$ where $m \in \mathbb{Z}$ and $n \in N_0$ was born on some finite day.
- On the other hand, every real number $x$ that is not this *dyadic rational* was born on day $\omega$, by taking as its left options all dyadic rationals smaller than it and as its right options all d. r. greater than it.

However, surreal numbers contain more than just real numbers and ordinals. Some examples:

- $\epsilon := \{0|1, \frac{1}{2}, \frac{1}{3}, \dots\}$. This number lies between *any* real number and zero.

Also, interestingly enough $\epsilon\omega = 1$, so $\epsilon = \frac{1}{\omega}$.

- $\omega - 1 = \{1, 2, 3, \dots|\omega\}$ is an infinite number less than $\omega$. So is e.g. $\frac{1}{2}\omega$ or $\sqrt{\omega}$ (yes, even square roots, exponentials et cetera can be defined for any surreal number).

Of course, there are many more interesting numbers (infinities and infinities...).

# Further reading

For a rigorous treatment of surreal numbers see e.g.

- Norman L. Alling, *Foundations of Analysis over Surreal Number Fields*, 1987