Variation on harmonic theme

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June 2021

Content

About me

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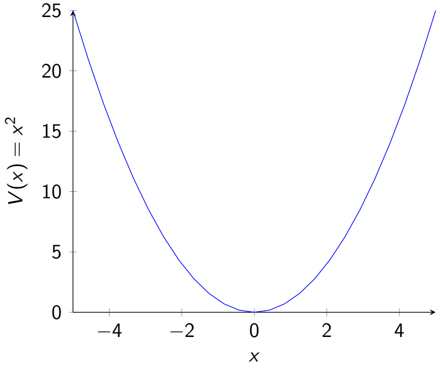
4. work: waiter for 7 years, manager in Covid testing center
5. Bachelor thesis - Variation on harmonic theme

Linear harmonic oscillator

1. model describing body on spring or pendulum
2. Elastic force: $F = -kx$; Potential: $V(x) = \frac{1}{2}kx^2$
3. $m\ddot{x} + kx = 0$; $\ddot{x} + \omega^2 x = 0$; $\omega = \sqrt{\frac{k}{m}}$
4. Solution: $x(t) = A\sin(\omega t + \varphi)$

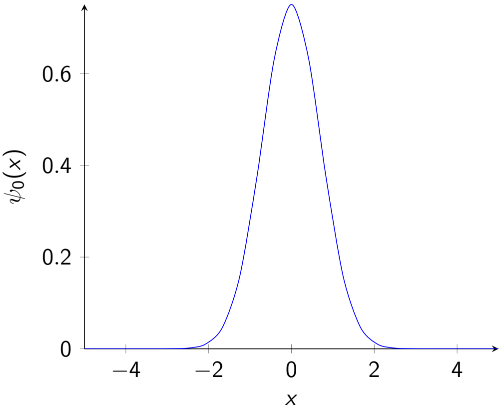


Quantum oscillator

1. Hilbert space: $\mathscr{H} = L^2(\mathbb{R}, dx)$
2. We set units, that $M = \hbar = 2$.
3. Same potential $V(x) = \omega^2 \hat{Q}^2$, now operator of multiplication on Hilbert space ($\hat{Q} = x\cdot$)
4. momentum p now also operator $\hat{P} = -2i\frac{\partial}{\partial x}$
5. Hamiltonian: $\hat{H}_{HO} = -\frac{d^2}{dx^2} + \omega^2 x^2$.
6. Hamiltonian is self-adjoint on $Dom(\hat{H}_{HO}) = \left\{\psi \in L^2(\mathbb{R}, dx); \psi, \psi' \text{ a.c.}, \hat{H}\psi \in \mathscr{H}\right\}$
7. Spectrum of Hamiltonian and eigenvectors?

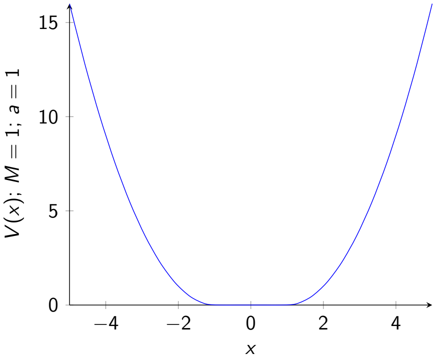
Quantum oscillator - solution

1. $\sigma(\hat{H}_{HO}) = \sigma_p(\hat{H}_{HO}) = \left\{ E_n = 2\omega(n + \frac{1}{2}) \mid n \in \mathbb{N}_0 \right\}$
2. Eigenvector in L^2: $\psi_n(x) = A_n H_n(x) e^{-\frac{\omega x^2}{2}}$
3. Ground state: $E_0 = \omega$: $\psi_0(x) = \frac{1}{\sqrt[4]{\pi}} e^{-\omega x^2}$



Plateau modification

1. Modified potential: $V(x) = \begin{cases} \omega^2(x+a)^2; x < -a \\ 0; |x| < a \\ \omega^2(x-a)^2; x > a \end{cases}$
2. $\hat{H} = \frac{d^2}{dx^2} + V(x)$
3. $\hat{H}$ is self-adjoint on the same domain as $\hat{H}_{HO}$ [Kato-Rellich]



Solution

1. Now we deal with Schrödinger equation in 3 seperate regions then connect (C^1) the solutions in points $-a$ and a

Quantization condition

$$\left[\frac{\omega}{\Gamma^2\left(\frac{1}{4}-\frac{E}{4\omega}\right)}-\frac{E}{4\Gamma^2\left(\frac{3}{4}-\frac{E}{4\omega}\right)}\right]\sin\left(2\sqrt{E}a\right)+\frac{\sqrt{E\omega}}{\Gamma\left(\frac{3}{4}-\frac{E}{4\omega}\right)\Gamma\left(\frac{1}{4}-\frac{E}{4\omega}\right)}\cos\left(2\sqrt{E}a\right)=0$$

2. From this equation we get allowed energy levels, values from Hamiltonian spectrum.

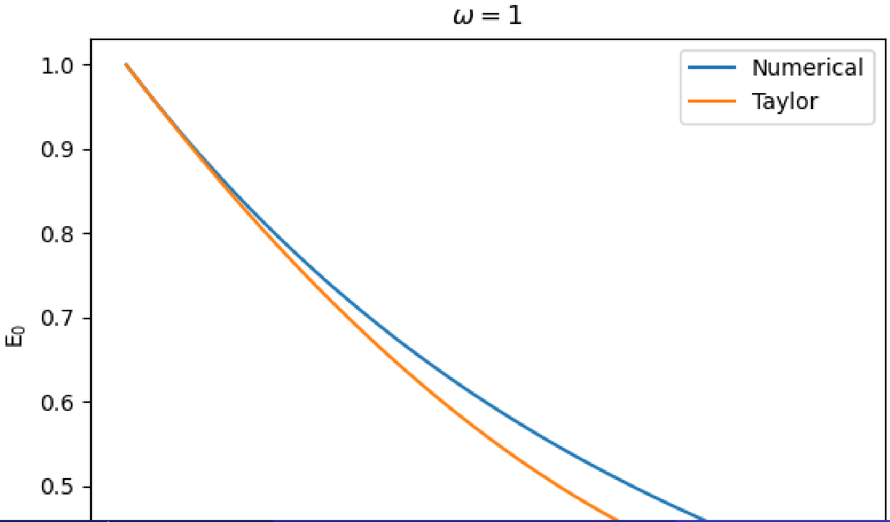
Perturbation theory

1. Investigation of this model for small values of a
    - by expanding the equation to Taylor series
    - by quantum perturbation theory
2. Both methods lead to the same result

1^{st} order perturbation theory for ground state energy

$$E_0(a) \sim \omega - \frac{2\omega\sqrt{\omega}}{\sqrt{\pi}}a$$

Numerical solution compared with perturbation theory



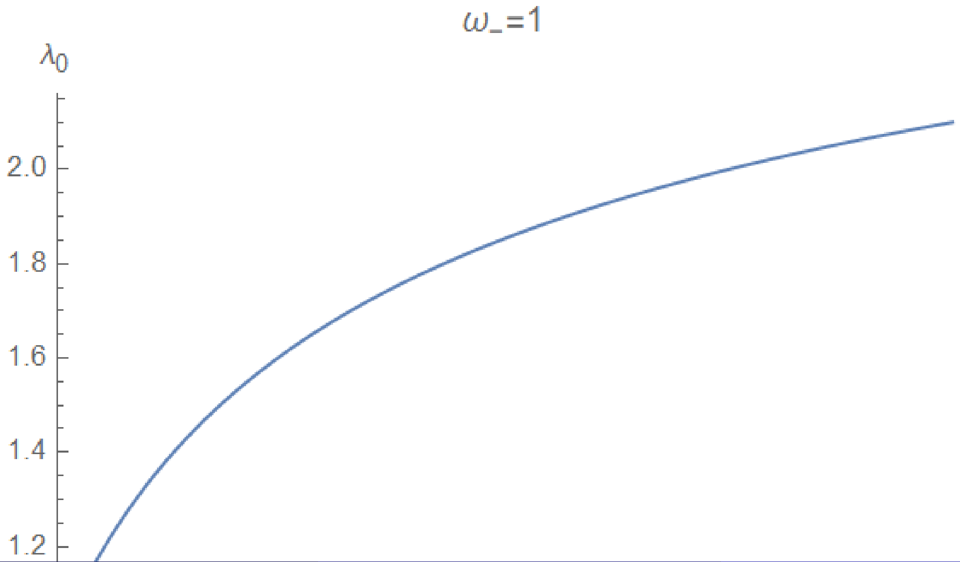
Different frequencies on half-axes modification

1. $\omega \neq$ const. $\rightarrow \omega(x) = \begin{cases} \omega_+; x \geq 0 \\ \omega_-; x < 0 \end{cases}$
2. Potential: $V(x) = \begin{cases} \omega_-^2 x^2; x < 0 \\ \omega_+^2 x^2; x > 0 \end{cases}$

Hamiltonian spectrum equation

$$\frac{\sqrt{\omega_+}}{2\Gamma(\frac{3}{4} - \frac{E}{4\omega_-})\Gamma(\frac{1}{4} - \frac{E}{4\omega_+})} + \frac{\sqrt{\omega_-}}{2\Gamma(\frac{3}{4} - \frac{E}{4\omega_+})\Gamma(\frac{1}{4} - \frac{E}{4\omega_-})} = 0$$

Ground state energy depending on ω_+



Intermezzo - Bloch theory

1. Hamiltonian with periodic potential

Bloch wave

$$\psi_q(x) = e^{iqx} u_q(x)$$

2. Eigenfunction of Hamiltonian is a complex exponential multiple of periodic function u_q with same period as potential, q is index of such functions, referred to quasi-momentum
3. Monodromy matrix $\mathbb{M}$ - period shift operator in base of linearly independent eigenvectors of Hamiltonian

Spectrum of Hamiltonian

$$\sigma(\hat{H}) = \{E \mid \left|\frac{1}{2} Tr\mathbb{M}\right| \leq 1\}$$

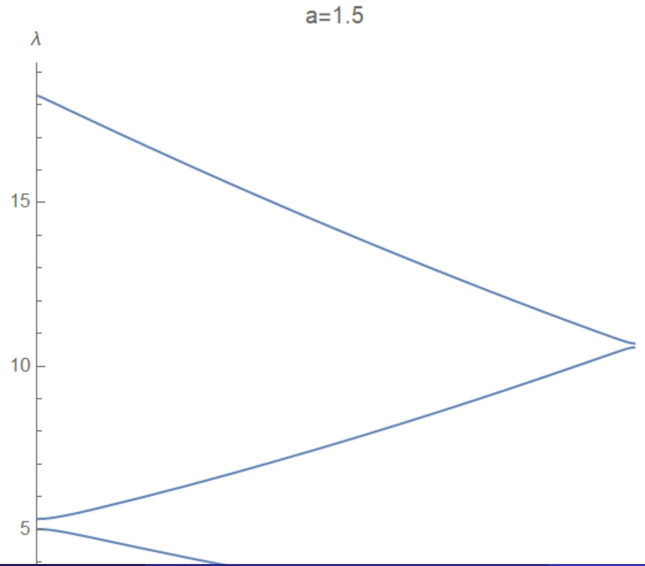
Periodic potential

1. Potential: $V(x) = \omega^2 x^2; |x| < a$
2. Periodically extended elsewhere
3. Bloch theory applied to this potential leads to condition

Spectral condition

$$\frac{a\,{}_1F_1\left(\frac{3}{4}-\frac{E}{4\omega},\frac{3}{2},\omega a^2\right)\left[-\omega\,{}_1F_1\left(\frac{1}{4}-\frac{E}{4\omega},\frac{1}{2},\omega a^2\right)+2\omega\left(\frac{1}{2}-\frac{E}{2\omega}\right)F_1\left(\frac{5}{4}-\frac{E}{4\omega},\frac{3}{2},\omega a^2\right)\right]}{{}_1F_1\left(\frac{1}{4}-\frac{E}{4\omega},\frac{1}{2},\omega a^2\right)\left[(1-\omega a)\,{}_1F_1\left(\frac{3}{4}-\frac{E}{4\omega},\frac{3}{2},\omega a^2\right)+a\left(\frac{1}{2}-\frac{E}{6\omega}\right){}_1F_1\left(\frac{7}{4}-\frac{E}{4\omega},\frac{5}{2},\omega a^2\right)\right]} \leq 0$$

Energy bands for periodic potential



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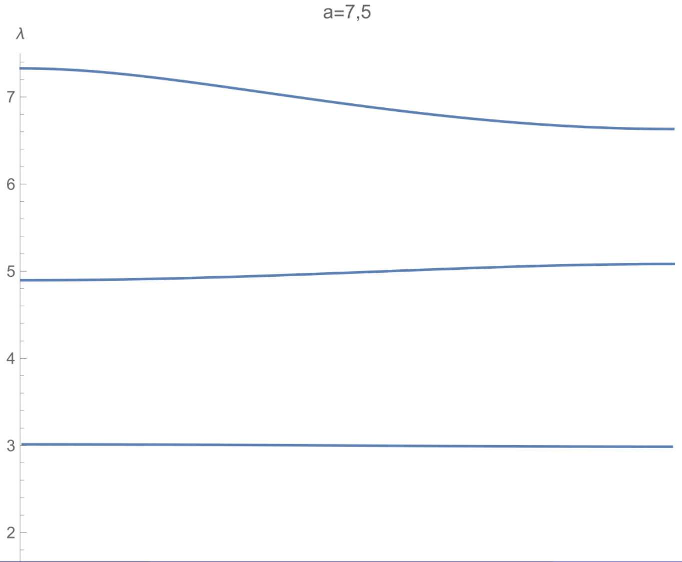

Conclusion

1. Results can be used in models of crystals in magnetic fields
2. Dielectric electrons in Lorenz interaction theory
3. Radiation in vacuum is described by set of linear harmonic oscillators