Clifford's Geometric Algebra in Differential Geometry

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Two well-known matrix algebras

- The Pauli Algebra

$$\{\sigma_i, \sigma_j\} = \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}$$

Matrix Clifford Algebras

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$$\{\sigma_i, \sigma_j\} = \sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij}$$

- The Dirac Algebra

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$$

Free tensor algebra

Definition

Let V be a vector space over field $\mathbb{F}$. For every $k \in \mathbb{N}$ we define the k-th tensor power of V as

$$T^k(V) = V \otimes V \otimes \ldots \otimes V,$$

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We also define the operation $\otimes_{k,l} : T^k(V) \times T^l(V) \to T^{k+l}(V)$

$$(v_1 \otimes \ldots \otimes v_k) \otimes_{k,l} (w_1 \otimes \ldots \otimes w_l) = v_1 \otimes \ldots \otimes v_k \otimes w_1 \otimes \ldots \otimes w_l,$$

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For scalars: $\lambda \otimes_{0,k} t = \lambda t = t \otimes_{k,0} \lambda$

Free tensor algebra

Definition

The free tensor algebra of a vector space over field $\mathbb{F}$ is the direct sum

$$T(V) = \bigoplus_{k=0}^{\infty} T^k(V)$$

equipped with associative multiplication $\otimes$ that is obtained from $\otimes_{k,l}$ via linear extension.

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Let $\mathcal{J}$ be a subspace of $T(V)$ such that

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The quotient space $T(V)/\mathcal{J}$ can be equipped with multiplication

$$[t] \cdot [t'] = [t \otimes t']$$

consistently. $(T(V)/\mathcal{J}, \cdot)$ is the quotient algebra of $T(V)$.

Clifford algebra

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Let V be a finite-dimensional real vector space equipped with a non-degenerate quadratic form Q. The Clifford algebra $Cl(V, Q)$ is the quotient algebra of $T(V)$ given by an ideal

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- Distributive over addition
- Square of a vector is a scalar
- Finite-dimensional

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$$A \times B = \frac{1}{2}(AB - BA)$$

Algebra of $\mathbb{E}^3$

Algebra is spanned by

$$\{1, e_1, e_2, e_3, B_1 \equiv e_2e_3, B_2 \equiv e_3e_1, B_3 \equiv e_1e_2, I \equiv e_1e_2e_3\}$$

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With our notation and the duality we can write

$$e_ie_j = \delta_{ij} + I\varepsilon_{ijk}e_k$$

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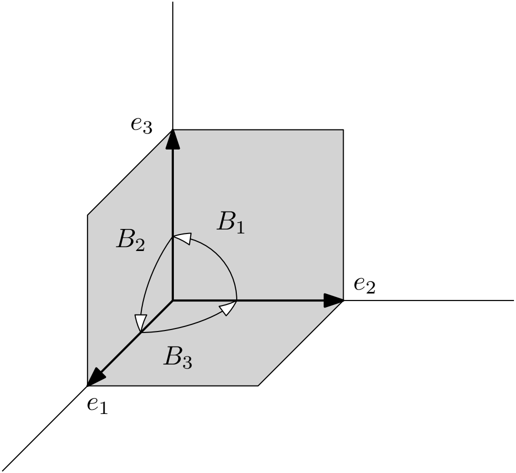

Rotations in two dimensions

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It is important to note that $(e_1 e_2)^2 = -1$, from that

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So the rotation is realized as

$$e^{-\varphi\, e_1e_2}(xe_1 + ye_2) = (x\cos\varphi + y\sin\varphi)e_1 + (-x\sin\varphi + y\cos\varphi)e_2$$

General rotations

Simple rotation in a plane associated with a unit bivector B acts only on the part parallel to the plane

$$e^{-\frac{\varphi}{2}B} v e^{\frac{\varphi}{2}B} = v_{\perp} + e^{-\varphi B} v_{\parallel}$$

General rotations are composed of simple rotations and the corresponding objects are called rotors

Pseudoscalars

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Subspaces of V generate subalgebras of $Cl(V, Q)$
One-to-one correspondence between unit blades and subspaces

Consider a submanifold $\mathcal{M}$ of $\mathbb{R}^N$ given by

$$f : \mathcal{R} \longrightarrow \mathbb{R}^N; \quad (u^1, \ldots, u^d) \mapsto f(u^1, \ldots, u^d)$$

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Its tangent space is spanned by $\left\{ f_\mu \equiv \frac{\partial f}{\partial u^\mu} \right\}_{\mu=1}^{d}$, these multiplied together give

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$I_{\mathcal{M}}$ is the unique d-blade corresponding to the tangent space

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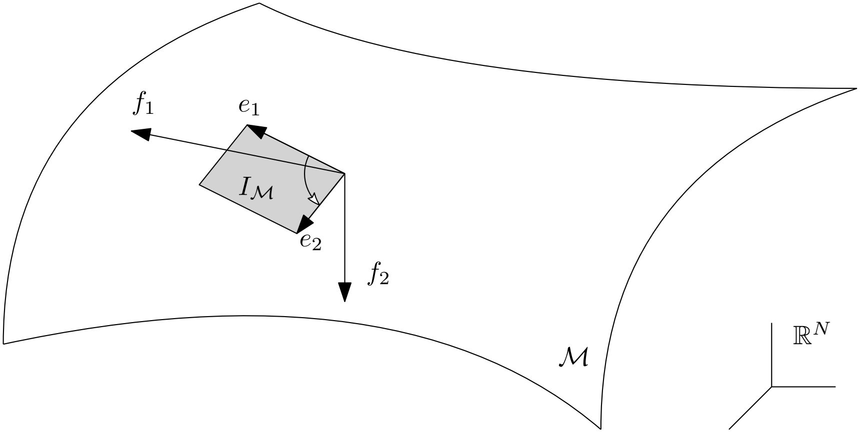

Shape operator

Derivative of $I_{\mathcal{M}}$

$$\partial_\mu I_{\mathcal{M}} = \partial_\mu (e_1 \wedge e_2 \wedge \ldots \wedge e_d) = I_{\mathcal{M}} \sum_{k=1}^{d} e_k \wedge P^{\perp}(\partial_\mu e_k)$$

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We define the shape operator as

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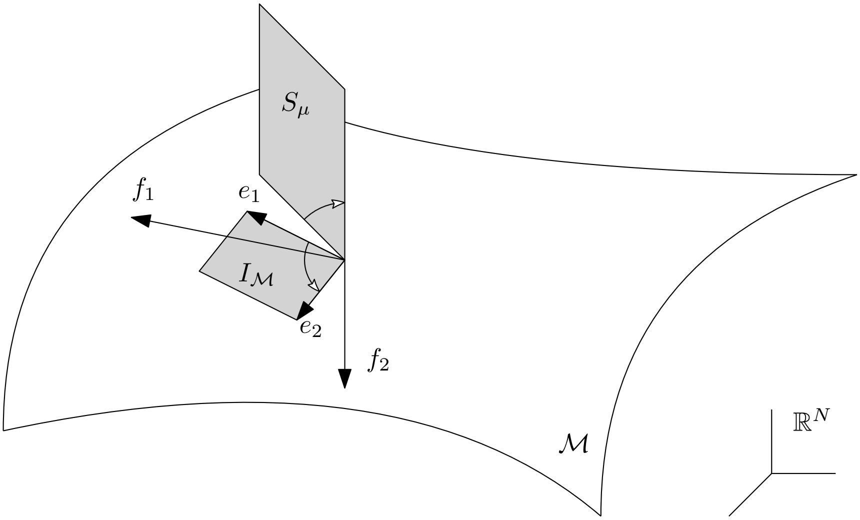
The expansion of $I_{\mathcal{M}}$

$$I_{\mathcal{M}}(u + \varepsilon c_\mu) = I_{\mathcal{M}}(u) + \varepsilon\, \partial_\mu I_{\mathcal{M}}(u) + \ldots \approx e^{-\frac{\varepsilon}{2} S_\mu(u)} I_{\mathcal{M}}(u) e^{\frac{\varepsilon}{2} S_\mu(u)}$$

Shape operator



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Shape operator encodes connection on $\mathcal{M}$

$$\Gamma^\rho{}_{\mu\nu} = e^\rho \cdot (\partial_\nu e_\mu + S_\nu \cdot e_\mu)\,.$$

Operator of covariant derivative in coordinate direction

$$
\begin{aligned}
D_\mu A(u) &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(e^{\frac{1}{2}\varepsilon S_\mu} A(\gamma^\mu(\varepsilon)) e^{-\frac{1}{2}\varepsilon S_\mu} - A(u) \right) \\
&= \partial_\mu A(u) - A(u) \times S_\mu .
\end{aligned}
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Shape operator is "torsion-free"

$$D_\mu S_\nu - D_\nu S_\mu = 0$$

Curvature

Commutator of covariant derivatives

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Curvature bivector

$$\Omega_{\mu\nu} = D_\mu S_\nu - D_\nu S_\mu + S_\mu \times S_\nu = S_\mu \times S_\nu$$

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- Curvature given algebraically

$$\Omega_{\mu\nu} = S_\mu \times S_\nu$$