On the pseudospectrum of the harmonic oscillator with imaginary cubic potential

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Metody algebry a funkcionální analýzy v aplikacích

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Based on:
On the pseudospectrum of the harmonic oscillator with imaginary cubic potential
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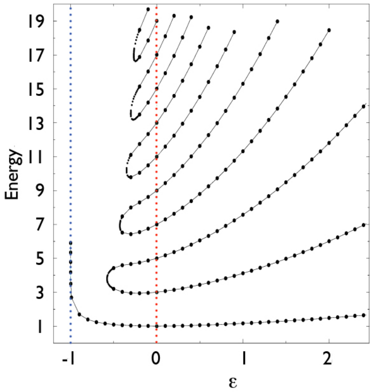
Outline of the talk

- $\mathcal{PT}$-symmetric quantum mechanic
  - Origins and some aspects of the theory
- The model
  - Introduction of the operator H
  - Known properties of H
- The pseudospectrum
  - Study of the pseudospectrum of H
- Consequences of the pseudospectral properties of H

Origins of $\mathcal{PT}$-symmetric Quantum mechanics

- Operator $-\Delta + \mathrm{i}x^3$ on $L^2(\mathbb{R})$ possesses real spectrum
  [Bender, Boettcher 98]

- More generally: $-\Delta + x^2(\mathrm{i}x)^\varepsilon$ for $\varepsilon > 0$



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? Due to $\mathcal{PT}$-symmetry ?

Operator H is $\mathcal{PT}$-symmetric

$$\Updownarrow$$

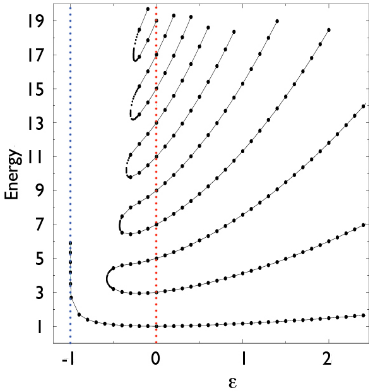
$[H, \mathcal{PT}] = 0$ (in operator sense)

- Parity

  $(\mathcal{P}\psi)(x) = \psi(-x)$

- Time reversal

  $(\mathcal{T}\psi)(x) = \overline{\psi(x)}$



Some aspects of $\mathcal{PT}$-symmetry

Quasi-self-adjoint operators:

[Scholtz, Geyer, Hahne 92]

H is q-s-a if there exists a positive bounded operator Θ with bounded inverse (called metric) such that $H^* = \Theta H \Theta^{-1}$

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- Change of Hilbert space
  - H is self-adjoint in Hilbert space $\left(L^2, \langle \cdot, \Theta \cdot \rangle\right)$
- Similarity to a self-adjoint operator
  - $h = \Theta^{1/2} H \Theta^{-1/2}$ is self-adjoint
  - solves problem with Stone's theorem

Physical relevance

- Suggestions
  - nuclear physics [Scholtz, Geyer, Hahne 92]
  - optics [Klaiman, Günther, Moiseyev 08], [Schomerus 10]
  - solid state physics [Bendix, Fleischmann, Kottos, Shapiro 09]
  - superconductivity [Rubinstein, Sternberg, Ma 07]
  - electromagnetism [Ruschhaupt, Delgado, Muga 05], [Mostafazadeh 09]
  - scattering [Hernandez-Coronado, Krejčiřík, Siegl 11]

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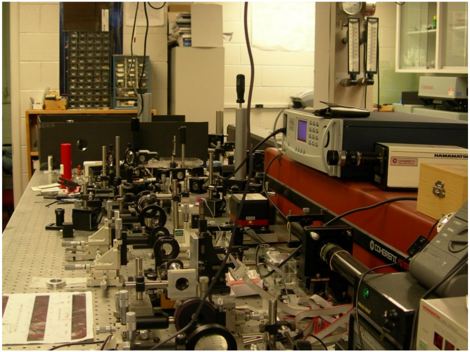
- Experiments
  - optics
    [Guo *et al.* 09],
    [Rüter *et al.* 10]



Introduction of the model

- Hilbert space $L^2(\mathbb{R})$

$$H := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2 + \mathrm{i}x^3,$$

$$\mathrm{Dom}(H) := \left\{ \psi \in L^2(\mathbb{R}) \;\middle|\; -\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + x^2\psi + \mathrm{i}x^3\psi \in L^2(\mathbb{R}) \right\}$$

- Observation of real spectrum
  [Caliceti, Graffi, Maioli 80]
  $\Rightarrow$ attributed to the $\mathcal{PT}$-symmetry

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  $\Rightarrow$ attributed to the $\mathcal{PT}$-symmetry

- Resembles the "Bender oscillator"

- Results hold for the more general case $x^3 \to x^{2n+1}$ $(n \geq 1)$

Properties of H

- $\mathrm{Dom}(H) = \left\{ \psi \in W^{2,2}(\mathbb{R}) \;\middle|\; x^3\psi \in L^2(\mathbb{R}) \right\}$ [Caliceti, Graffi, Maioli 80]
- H is closed [Caliceti, Graffi, Maioli 80]
- H is an operator with compact resolvent [Caliceti, Graffi, Maioli 80]
  $\Rightarrow \sigma(H)$ consists of isolated eigenvalues of finite algebraic multiplicity
- Eigenvalues of H are real and simple [Shin 02]
- H is m-accretive
  $\Rightarrow \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda < 0\} \subset \rho(H)$
- H is $\mathcal{PT}$-symmetric
- Resolvent is a Hilbert-Schmidt operator [Caliceti, Graffi, Maioli 80]

Results about $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mathrm{i}x^3$

- All of the properties of H [Caliceti, Graffi, Maioli 80],[Dorey, Dunning, Tateo 01], [Tai 05]
- Resolvent is a trace class operator [Mezincescu 01]

Results about $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mathrm{i}x^3$

- All of the properties of H [Caliceti, Graffi, Maioli 80],[Dorey, Dunning, Tateo 01], [Tai 05]
- Resolvent is a trace class operator [Mezincescu 01]

Recent results:

- Completeness of eigenfunctions in $L^2(\mathbb{R})$ [Siegl, Krejčiřík 12]
- Existence of a bounded metric operator Θ [Siegl, Krejčiřík 12]
- Θ cannot have bounded inverse [Siegl, Krejčiřík 12]
- Wild pseudospectral behaviour [Krejčiřík, Siegl, Tater, Viola 14]

Does something similar hold for H?

Definition of pseudospectrum

... in the previous lecture

Pseudospectral behaviour

Trivial $\times$ Non-trivial

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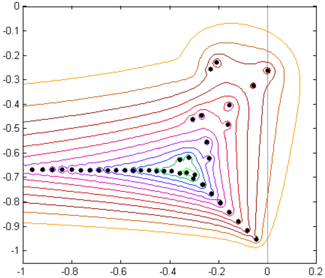
Trivial $\times$ Non-trivial

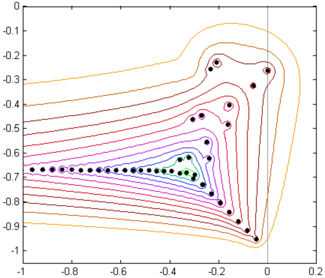


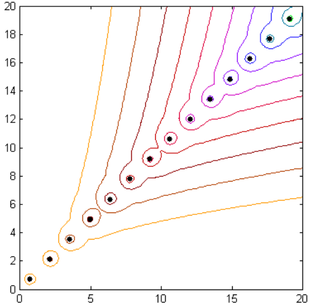
$\mathrm{i}\alpha T^{-1}\left(\frac{1}{\mathrm{i}\alpha R}T^2-(1-x^2)T^2-2\right)$, where
$T=\frac{\mathrm{d}^2}{\mathrm{d}x^2}-\alpha^2$, Ors-Sommerfeld operator

For some $C>0$ holds $\sigma_\varepsilon(A)\subset\{z\in\mathbb{C}\mid \mathrm{dist}(z,\sigma(A))<C\varepsilon\}$

Pseudospectral behaviour

Trivial $\times$ **Non-trivial**





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$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\mathrm{i}x^2$, Davies' oscillator

For some $C>0$ holds $\sigma_\varepsilon(A)\subset\{z\in\mathbb{C}\mid \mathrm{dist}(z,\sigma(A))<C\varepsilon\}$

$\sigma_\varepsilon(A)$ is not in any neighbourhood of $\sigma(A)$

Pseudospectrum of H

Idea: Use semiclassical analysis [Davies 99]

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2 + \mathrm{i}x^3$$

Unitary transformation

$$(U\psi)(x) := \tau^{1/2}\,\psi(\tau x)$$

leads to the semiclassical analogue of H:

$$UHU^{-1} = \tau^3 H_h,$$

where

$$H_h := -h^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + h^{2/5}x^2 + \mathrm{i}x^3$$

and $h := \tau^{-5/2}$ is the semiclassical parameter.

How to study the pseudospectrum of H_h ?

Semiclassical technique

Theorem:
Let

$$T_h := -h^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_h(x),$$

where V_h are analytic potentials in x for all $h > 0$ small enough which take the form $V_h(x) = V_0(x) + \widetilde{V}(x,h)$, where $\widetilde{V}(x,h) \to 0$ locally uniformly as $h \to 0$.

Let λ be from the set

$$\Lambda_h := \left\{ \xi^2 + V_h(x) \,\middle|\, (x,\xi) \in \mathbb{R}^2, \xi \operatorname{Im} V_h'(x) < 0 \right\},$$

where the dash denotes standard differentiation with respect to x in $\mathbb{R}$.

Then there exists some $C = C(\lambda) > 1$, some $h_0 = h_0(\lambda) > 0$, and an h-dependent family of $C_c^\infty(\mathbb{R})$ functions $\{\psi_h\}_{0<h\leq h_0}$ with the property that, for all $0 < h \leq h_0$,

$$\|(T_h - \lambda)\psi_h\| < C^{-1/h}\|\psi_h\|.$$

Remarks to the Theorem

- Analogue of a result in [Davies 99], [Dencker, Sjöstrand, Zworski 04] for potential depending on h
- Proof inspired by [Krejčiřík, Siegl, Tater, Viola 14]
- For $\lambda \in \Lambda$ we have $\lambda \in \sigma_\varepsilon(H_h)$ for all $\varepsilon \geq C(\lambda)^{-1/h}$

Nomenclature:

- $f(x, \xi) := \xi^2 + V_h(x)$ – the symbol associated with A_h
- Closure of Λ – the semiclassical pseudospectrum

Ingredients of the proof

1. Fix $\lambda = x_0^2 + V_h(x_0)$
2. JWKB approximation of the solution to $(T_h - \lambda)u = 0$

$$u(x,h) := \mathrm{e}^{\mathrm{i}\phi(x,h)/h} \sum_{j=0}^{N(h)} h^j a_j(x,h),$$

3. Eikonal equation

$$\begin{aligned} f(x, \phi'(x,h)) - \lambda &= 0 \\ \phi'(x,h)^2 + V_h(x) - \lambda &= 0 \\ \phi'(x,h) &= \pm\sqrt{\lambda - V_h(x)} \end{aligned}$$

The solution is analytic and well-defined near x_0:

$$\phi(x,h) = -\mathrm{sgn}\left(\mathrm{Im}\, V_h'(x_0)\right) \int_0^x \sqrt{\lambda - V_h(y)}\, \mathrm{d}y.$$

Ingredients of the proof

4. Transport equation

$$\mathrm{e}^{-\mathrm{i}\phi/h}\,(T_h-\lambda)\,\mathrm{e}^{\mathrm{i}\phi/h}=\frac{2h}{\mathrm{i}}\left(\phi'\frac{\mathrm{d}}{\mathrm{d}x}+\frac{1}{2}\phi''\right)-h^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}$$

If we in $a(x,h):=\sum_{j=0}^{N(h)} h^j a_j(x,h)$ set

$$\phi'(x,h)a_0'(x,h)+\frac{1}{2}\phi''(x,h)a_0(x,h)=0,$$
$$\phi'(x,h)a_j'(x,h)+\frac{1}{2}\phi''(x,h)a_j(x,h)=\frac{\mathrm{i}}{2}a_{j-1}''(x,h)$$

then $\mathrm{e}^{-\mathrm{i}\phi/h}\,(T_h-\lambda)\,\mathrm{e}^{\mathrm{i}\phi/h}a(x,h)=-h^{N+2}a_N''(x,h)$.

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$$\begin{aligned}\phi'(x,h)a_0'(x,h)+\frac{1}{2}\phi''(x,h)a_0(x,h)&=0,\\ \phi'(x,h)a_j'(x,h)+\frac{1}{2}\phi''(x,h)a_j(x,h)&=\frac{\mathrm{i}}{2}a_{j-1}''(x,h)\end{aligned}$$

then $\mathrm{e}^{-\mathrm{i}\phi/h}\,(T_h-\lambda)\,\mathrm{e}^{\mathrm{i}\phi/h}a(x,h)=-h^{N+2}a_N''(x,h)$.We add boundary conditions $a_0(x_0,h)=1$ and $a_j(x_0,h)=0$ for $j>0$ and get analytic and well-defined solution near x_0:

$$\begin{aligned}a_0(x,h)&=\frac{\sqrt{\phi'(x_0,h)}}{\sqrt{\phi'(x,h)}},\\ a_j(x,h)&=\frac{1}{\sqrt{\phi'(x_0,h)}}\int_0^x\frac{\mathrm{i}\,a_{j-1}''(y,h)}{2\sqrt{\phi'(y,h)}}\,\mathrm{d}y.\end{aligned}$$

Ingredients of the proof

5. $|a_j(x,h)| \leq C_1^{j+1} j^j$

   $\Rightarrow$ $a(x,h) := \sum_{0 \leq j \leq (\mathrm{e} C_1 h)^{-1}} h^j a_j(x,h)$ is uniformly bounded analytic function

6. Pseudomode $\psi_h(x) := \mathrm{e}^{\mathrm{i}\phi(x,h)/h} \chi(x) a(x,h)$,

   - $\chi \in C_c^\infty(\mathbb{R})$ identically equal to 1 in some neighbourhood of x_0 and with sufficiently small support

7. $\|(T_h - \lambda)\psi_h\| \leq C\mathrm{e}^{-1/h}$

8. $\|\psi_h\|$ not exponentially small for $h \to 0$

Application on H_h

$$H_h = -h^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + h^{2/5} x^2 + \mathrm{i}x^3$$

- $V_0(x) = \mathrm{i}x^3,\ V_h(x) = h^{2/5} x^2$
- $\Lambda \supset \left\{ \lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > 0, |\arg\lambda| < \arctan(\sqrt{\operatorname{Re}\lambda}) \right\}$
- for $\lambda \in \Lambda$ the Theorem gives $(\tau = h^{2/5})$

$$\left\| \left(H - \tau^3 \lambda \right)^{-1} \right\| = \tau^{-3} \left\| (H_h - \lambda)^{-1} \right\| > h^{6/5} C(\lambda)^{1/h}$$

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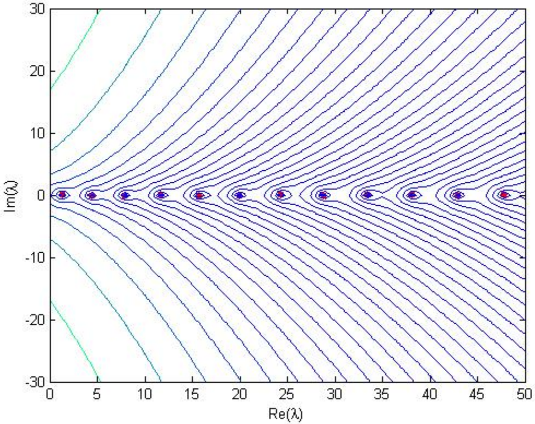
$$\Downarrow$$

For $\delta > 0$ exist constants $C_1, C_2 > 0$ such that for all $\varepsilon > 0$ small, σ_ε contains the set

$$\left\{ \lambda \in \mathbb{C} \;\middle|\; |\lambda| > C_1,\ |\arg\lambda| < \arctan\sqrt{\operatorname{Re}\lambda} - \delta,\ |\lambda| \geq C_2 \left(\log \frac{1}{\varepsilon} \right)^{6/5} \right\}$$

Numerical visualisation



- Spectrum (red dots) and ε-pseudospectra (enclosed by blue-green lines)
- $\varepsilon = 10^{-7}(\text{blue}), 10^{-6.75}, 10^{-6.5}, \ldots, 10^{1}(\text{green})$
- Used computational method can be found in [Trefethen 00].

Basis properties

Let us denote by $\{\psi_k\}_{k=1}^{+\infty}$ the set of eigenfuncions of H

- The eigenfunctions of H form a complete set in $L^2(\mathbb{R})$
  (i.e. span of ψ_k is dense in $L^2(\mathbb{R})$)
  - resolvent is trace class operator (shown using [Almog, Helffer 14])

  $\Rightarrow$ completeness of its eigenfunctions

  $\Rightarrow$ Spectral mapping theorem

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- The eigenfunctions of H do not form a (Schauder) basis in $L^2(\mathbb{R})$
  (Schauder basis – every $\psi \in L^2(\mathbb{R})$ can be uniquely expressed as $\psi = \sum_{k=1}^{+\infty} \alpha_k \psi_k$, where $\alpha_k \in \mathbb{C}$)
  - $\|(H - \lambda)^{-1}\|$ grows exponentially fast for $|\lambda|$ large

  $\Rightarrow$ spectral projections cannot be polynomially bounded [Davies 00]

  $\Rightarrow$ $\{\psi_k\}_{k=1}^{+\infty}$ cannot form a basis

Consequences of the non-trivial pseudospectrum

- H is not similar to a self-adjoint operator via bounded and boundedly invertible transformation Ω
  - If H were similar to a self-adjoint h then

    $$\sigma_{\varepsilon/\kappa}(H) \subset \sigma_\varepsilon(h) \subset \sigma_{\varepsilon\kappa}(H),$$

    where $\kappa = \|\Omega\|\|\Omega^{-1}\|$

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- $-\mathrm{i}H$ is not a generator of a bounded semigroup
  - Exponential growth of $\|(H-\lambda)^{-1}\|$
  - Result follows from [Davies 07]

Conclusions

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2 + \mathrm{i}x^3$$

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Results:

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- Wild behaviour of the pseudospectrum

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Moral: Pseudospectrum reveals what spectrum hides.

Thank you for your attention!