Remarks on the convergence of pseudospectra

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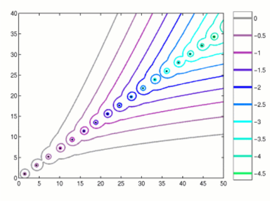
Based on

[1] S. Bögli and P. Siegl: *Remarks on the convergence of pseudospectra*, Integral Equations and Operator Theory 80, 2014, 303-321, arXiv:1408.3431.

[2] S. Bögli, P. Siegl, and C. Tretter: *Approximations of spectra of Schrödinger operators with complex potentials on* $\mathbb{R}^d$, 32 pp.

1. Constant resolvent norm
2. Convergence of pseudospectra & applications to Schrödinger operators

Rotated oscillator[1]



$A := -\partial_x^2 + \mathrm{i}x^2$ in $L^2(\mathbb{R})$

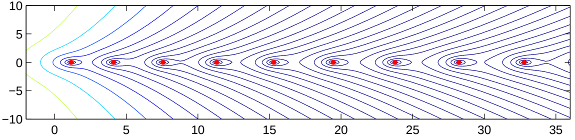
$\sigma(A) = \left\{ e^{\mathrm{i}\pi/4}(2k+1) : k = 0, 1, 2, \dots \right\}$

$\sigma_\varepsilon(A)$ much larger than ε-neighborhood of $\sigma(A)$

Imaginary cubic oscillator[2]

$A := -\partial_x^2 + \mathrm{i}x^3$ in $L^2(\mathbb{R})$

$\sigma(A)$ is discrete and real



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[1] L. Boulton. *J. Operator Theory* 47 (2002), pp. 413–429; E. B. Davies. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* 455 (1999), pp. 585–599; P. Exner. *J. Math. Phys.* 24 (1983), pp. 1129–1135; K. Pravda-Starov. *J. London Math. Soc.* 73 (2006), pp. 745–761.

[2] C. M. Bender and S. Boettcher. *Phys. Rev. Lett.* 80 (1998), pp. 5243–5246; D. Krejčiřík et al. arXiv:1402.1082. 2014; P. Siegl and D. Krejčiřík. *Phys. Rev. D* 86 (2012), 121702(R).

Rotated oscillator

- operator:

$$A := -\partial_x^2 + \mathrm{i}x^2 \text{ in } L^2(\mathbb{R})$$

- spectrum:

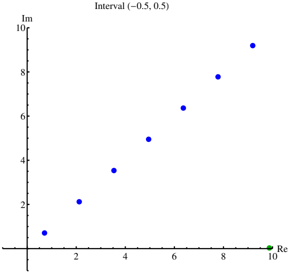
$$\sigma(A) = \left\{ e^{\mathrm{i}\pi/4}(2k+1) : k = 0, 1, 2, \dots \right\}$$

Domain truncation

- sequence of operators:

$$A_n := -\partial_x^2 + \mathrm{i}x^2 \text{ in } L^2((-n,n)) + \text{Dirichlet BC at } \pm n$$

- does $\sigma(A_n) \to \sigma(A)$?



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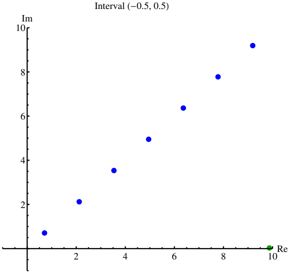

- let A be a closed densely defined operator in a Banach space $\mathcal{X}$

Pseudospectrum

$$\sigma_\varepsilon(A) := \sigma(A) \cup \left\{ z \in \mathbb{C} \ : \|(A-z)^{-1}\| > \frac{1}{\varepsilon} \right\}$$

Another definition with the non-strict inequality $\geq$

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The question

- Does $\Sigma_\varepsilon(A) = \overline{\sigma_\varepsilon(A)}$ hold?

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The question differently

Let $M > 0$ and $A \in \mathcal{C}(\mathcal{X})$.

Can $\{z \in \rho(A) : \|(A - z)^{-1}\| = M\}$ have an open subset in $\mathbb{C}$?

Resolvent as a holomorphic function

- recall: $(A - z)^{-1}$ is a holomorphic function on $\rho(A)$
- use the maximum modulus principle?

Maximum modulus principle

Let f be holomorphic on a connected open subset Ω of $\mathbb{C}$. Let $z_0 \in \Omega$ and $|f(z)| \leq |f(z_0)|$ for all z in a neighborhood of z_0. Then f is constant on Ω.

Holomorphic matrix-valued function

$$A(z) = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$$

- $\|A(z)\| = 1$ for $|z| \leq 1$
- but $(A - z)^{-1}$ is a very special function...

Constant resolvent norm - maximum modulus principle

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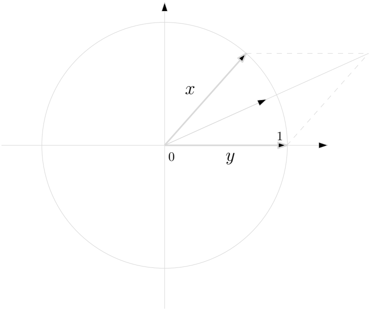
Uniformly convex Banach space

A Banach space $\mathcal{X}$ is uniformly convex, if for every $\varepsilon > 0$ exists $\delta > 0$ such that for all $x, y \in \mathcal{X}$ with $\|x\| = \|y\| = 1$:

$$\|x - y\| \geq \varepsilon \quad \Longrightarrow \quad \Big\| \frac{1}{2}(x + y) \Big\| \leq 1 - \delta$$

- geometrically: the unit ball is "uniformly round"



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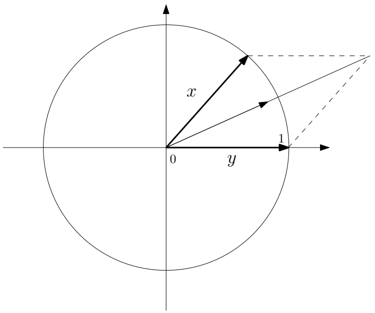

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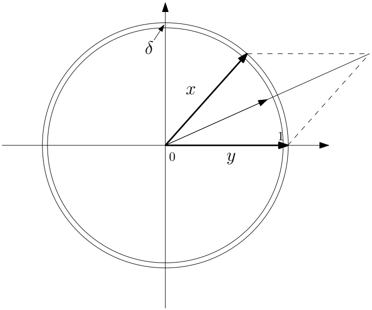

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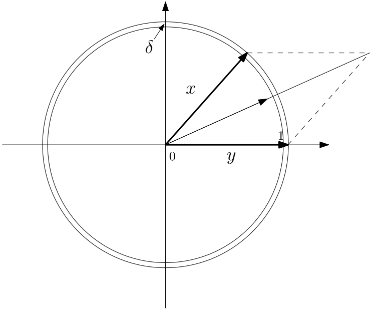

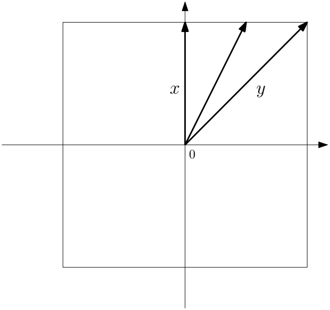

- any Hilbert space is uniformly convex (polarization identity)

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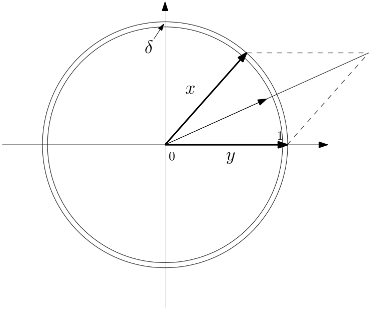

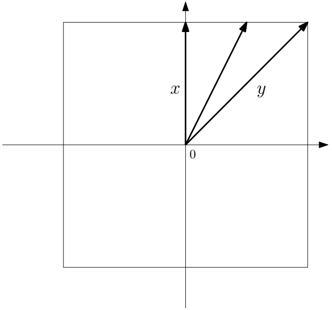

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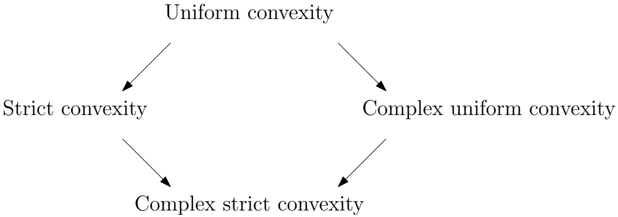
Various other convexity



Known results

- $\mathcal{X}$ a Banach space and $A \in \mathcal{C}(\mathcal{X})$
- $\lambda \mapsto \|(A-\lambda)^{-1}\|$ cannot be constant on an open subset $\Omega \subset \rho(A)$ if

  i) Globevnik[3]: $A \in \mathscr{B}(\mathcal{X})$ and Ω belongs to the unbounded component of $\rho(A)$

  ii) $A \in \mathscr{B}(\mathcal{X})$
  - Globevnik[6] if $\mathcal{X}$ is complex uniformly convex (*e.g.* Hilbert space, L^p-space with $1 \le p < \infty$)
  - Daniluk (1994) for Hilbert spaces
  - Böttcher-Grudsky-Silbermann[4] for L^p-spaces with $1 < p < \infty$
  - Harrabi[5] if $\mathcal{X}$ finite-dimensional
  - Shargorodsky[6] if $\mathcal{X}$ or $\mathcal{X}^*$ is complex uniformly convex (covers also $p = \infty$)

  iii) A generates a C_0 semigroup
  - Shargorodsky[7] if $\mathcal{X}$ or $\mathcal{X}^*$ is complex uniformly convex

  iv) A has compact resolvent
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    - Davies-Shargorodsky[8] if $\mathcal{X}$ or $\mathcal{X}^*$ is complex strictly convex (*e.g.* if $\mathcal{X}$ or $\mathcal{X}^*$ complex uniformly convex)

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[6] E. Shargorodsky. *Bull. Lond. Math. Soc.* 40 (2008), pp. 493–504.
[7] E. Shargorodsky. *Bull. Lond. Math. Soc.* 42 (2010), pp. 1031–1034.
[8] E. B. Davies and E. Shargorodsky. *Mathematika* online first (2015).

Known results

- $\mathcal{X}$ a Banach space and $A \in \mathcal{C}(\mathcal{X})$
- $\lambda \mapsto \|(A-\lambda)^{-1}\|$ cannot be constant on an open subset $\Omega \subset \rho(A)$ if
  i) Globevnik[3]: $A \in \mathscr{B}(\mathcal{X})$ and Ω belongs to the unbounded component of $\rho(A)$
  ii) $A \in \mathscr{B}(\mathcal{X})$
    - Globevnik[6] if $\mathcal{X}$ is complex uniformly convex (*e.g.* Hilbert space, L^p-space with $1 \le p < \infty$)
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Example with constant resolvent norm[9]

- $\alpha_k := k + 1$ and $\beta_k := 1 + 1/\alpha_k$, $k \in \mathbb{N}$
- 2×2 blocks
$$B_k := \begin{pmatrix} 0 & \alpha_k \\ \beta_k & 0 \end{pmatrix}, \quad k \in \mathbb{N},$$
- operator in $\ell^2(\mathbb{N})$: $A := \operatorname{diag}(B_1, B_2, B_3, \dots)$
- $\sigma(A) = \cup_{k\in\mathbb{N}} \sigma(B_k) = \{\pm\sqrt{k+2} : k \in \mathbb{N}\}$
- inverse of the block
$$(B_k - \lambda)^{-1} = \frac{1}{\alpha_k \beta_k - \lambda^2} \begin{pmatrix} \lambda & \alpha_k \\ \beta_k & \lambda \end{pmatrix}$$
- for $|\lambda| < 1$:
$$\lim_{k\to\infty} \|(B_k - \lambda)^{-1}\| = \left\| \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\| = 1$$
- for $|\lambda| < 1/2$:
$$\|(B_k - \lambda)^{-1}\| \le \frac{1}{\alpha_k \beta_k - |\lambda|^2} \left(\left\| \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right\| + \left\| \begin{pmatrix} 0 & \alpha_k \\ \beta_k & 0 \end{pmatrix} \right\| \right)$$
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Shargorodsky example

Example with constant resolvent norm

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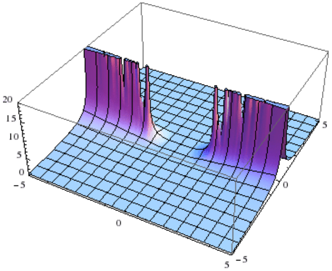
$$\|A\| = \sup_{k \in \mathbb{N}} \|(B_k - \lambda)^{-1}\| = 1$$

Numerics



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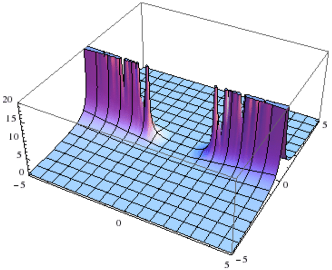

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Theorem [S. Bögli & PS, 2014]

Let $\mathcal{X}$ be a complex uniformly convex Banach space, $A \in \mathcal{C}(\mathcal{X})$. If there exist an open subset $\Omega \subset \rho(A)$ and a constant $M > 0$ such that

$$\|(A-\lambda)^{-1}\| = M, \quad \lambda \in \Omega,$$

then

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Sketch of the proof

- $F(\lambda) := (A-\lambda)^{-1}$ is analytic function with $\|F(\cdot)\| \equiv M$ on Ω
- take $\lambda_0 \in \Omega$ and $\{e_k\}_k \subset \mathcal{H}$ with $\|e_k\| = 1$ and $\|(A-\lambda_0)^{-1}e_k\| \to M$.
- Globevnik & Vidav[10]: $\|F'(\lambda_0)e_k\| \to 0$
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Corrolaries

i) If there exists a path $\gamma : [0, \infty) \to \rho(A)$ such that

$$\lim_{s \to \infty} |\gamma(s)| = \infty, \quad \lim_{s \to \infty} \|(A - \gamma(s))^{-1}\| = 0,$$

then resolvent norm cannot be constant on any open subset of $\rho(A)$.

ii) This applies if $A \in \mathscr{B}(\mathcal{X})$ since

$$\|(A - \lambda)^{-1}\| \leq (|\lambda| - \|A\|)^{-1}, \quad |\lambda| > \|A\|.$$

iii) This applies if A generates a C_0 semigroup since, by Hille-Yosida Theorem,

$$\exists\, C > 0,\, \omega \in \mathbb{R} : \quad \|(A - \lambda)^{-1}\| \leq C(\lambda - \omega)^{-1}, \quad \lambda \in (\omega, +\infty).$$

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Example - various resolvent norm behaviors

Operator matrix

$$T := \begin{pmatrix} 0 & f(A) \\ A & 0 \end{pmatrix} \quad \text{in} \quad \mathcal{H} \oplus \mathcal{H}$$

- $A = A^* > 0$ in $\mathcal{H}$ with discrete spectrum, $f : \mathbb{R} \to \mathbb{R}$ continuous
- for $A = \Delta$ in $L^2(\mathbb{R}^d)$ and $f(x) = 1$: T is the generator of wave equation

a) $\lim_{x \to +\infty} f(x) = 0 \implies \rho(T) = \emptyset$

b) $\lim_{x \to +\infty} f(x) = C > 0$ and $f(x) \geq C \implies$ constant $\|(T - z)^{-1}\|$ on $\Omega \subset \rho(T)$

- Shargorodsky example: $A = \mathrm{diag}(2, 3, 4, \dots)$ and $f(x) = 1 + 1/x$

c) $f(x) = |x|^\beta$, $\beta \in (0, 1) \implies \|(T - re^{i\phi})^{-1}\| = \mathcal{O}(r^{-2\beta/(\beta+1)})$ if $\phi \notin \{0, \pi\}$.

- decay $\implies \|(T - z)^{-1}\|$ is not constant on any open set
- decay not sufficient to generate a C_0 semigroup

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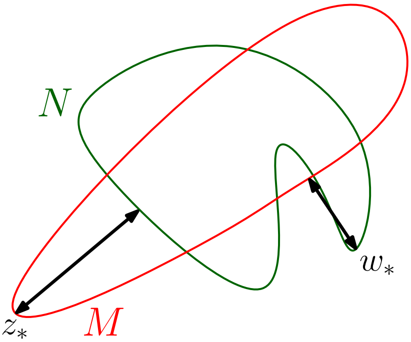
Hausdorff distance

- $M, N \subset \mathbb{C}$ non-empty and compact

$$d_{\mathrm{H}}(M,N) = \max\left\{ \max_{z\in M} \operatorname{dist}(z,N), \max_{w\in N} \operatorname{dist}(w,M) \right\}$$



Theorem [S. Bögli & PS, 2014]

Let

- $\mathcal{H}$ and $\mathcal{H}_n$, $n \in \mathbb{N}$, subspaces of a Hilbert space $\mathcal{H}_0$
- $A \in \mathcal{C}(\mathcal{H})$, $A_n \in \mathcal{C}(\mathcal{H}_n)$ densely defined
- $K \subset \mathbb{C}$ compact and $\varepsilon > 0$

If

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then

$$d_{\mathrm{H}}\left(\overline{\sigma_\varepsilon(A_n)} \cap K, \overline{\sigma_\varepsilon(A)} \cap K\right) \to 0, \quad n \to \infty .$$

Remarks

- previous result by Hansen (PhD thesis, 2008), problems on ∂K
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Operator

$$T = -\Delta + Q \quad \text{in} \quad L^2(\mathbb{R}^d)$$

- Q is complex valued and such that T has compact resolvent

Approximations

$$T_n = -\Delta + Q \quad \text{in} \quad L^2(\Omega_n)$$

- $\{\Omega_n\}_n$ are expanding bounded Lipschitz domains that exhaust $\mathbb{R}^d$; *e.g.*

$$\Omega_n = B_n(0), \quad n \in \mathbb{N}$$

- Dirichlet, Neumann or Robin BC are imposed on $\partial\Omega_n$

Questions

- Does $\sigma_\varepsilon(T_n)$ converge to $\sigma_\varepsilon(T)$?
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Rotated oscillator

- operator:

$$A := -\partial_x^2 + \mathrm{i}x^2 \text{ in } L^2(\mathbb{R})$$

- spectrum:

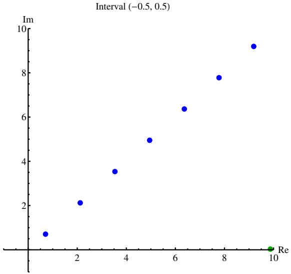
$$\sigma(A) = \left\{ e^{\mathrm{i}\pi/4}(2k+1) : \ k = 0, 1, 2, \ldots \right\}$$

Domain truncation

- sequence of operators:

$$A_n := -\partial_x^2 + \mathrm{i}x^2 \text{ in } L^2((-n,n)) \ + \ \text{Dirichlet BC at } \pm n$$

- $\Omega_n = (-n, n)$



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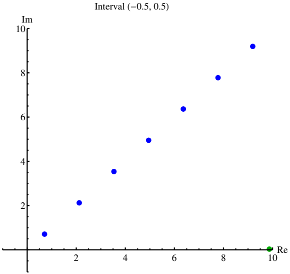

m-sectorial case

- 1D example: $Q(x) = (1 + \mathrm{i})x^2 + \mathrm{i}\delta(x)$
- decomposition: $Q = Q_0 + W$
  1. sectoriality: $L^1_{\mathrm{loc}}(\mathbb{R}^d) \ni Q_0$ has values in a sector with semi-angle $< \pi/2$
  2. growth at ∞: $|Q_0(x)| \to \infty$ as $|x| \to \infty$
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non-m-sectorial case

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Generalized norm resolvent convergence

$$\left\|(T_n-\lambda)^{-1}\chi_{\Omega_n}-(T-\lambda)^{-1}\right\| \to 0, \quad \lambda\in\rho(T).$$

Pseudospectral convergence

$$d_{\mathrm{H}}\left(\overline{\sigma_\varepsilon(A_n)}\cap K, \overline{\sigma_\varepsilon(A)}\cap K\right)\to 0, \quad n\to\infty.$$

Spectral convergence (spectral exactness)

1. Every eigenvalue λ of T is approximated:
   there is $\{\lambda_n\}_n$, $\lambda_n\in\sigma(T_n)$, such that $\lambda_n\to\lambda$ as $n\to\infty$.
2. Every accumulation point of $\{\lambda_n\}_n$ is an eigenvalue of T :
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Convergence rate for eigenvalues

- $\lambda\in\sigma(T)$ simple & ϕ is the corresponding eigenfunction

Then, there is $C>0$ such that

$$|\lambda-\lambda_n|\le C\|\phi\restriction\mathbb{R}^d\setminus B_n(0)\|.$$

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Examples

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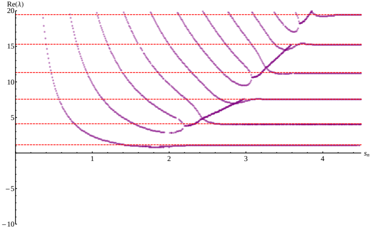
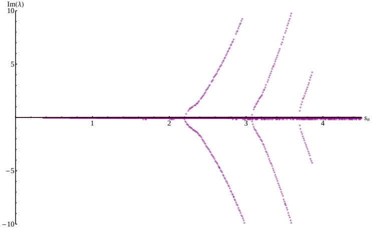
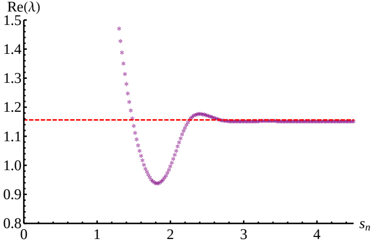
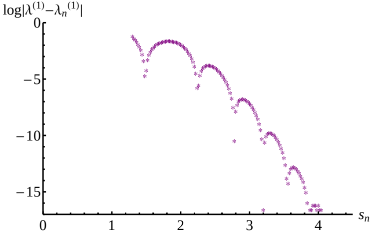
- $\sigma(T) \subset \mathbb{R}$





- the first eigenvalue and the rate (Dirichlet BC)





Examples

$$T = -\partial_x^2 + \mathrm{i}x, \quad \mathrm{Dom}\,(T) = W^{2,2}(\mathbb{R}) \cap \mathrm{Dom}\,(x)$$

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- all eigenvalues escape to infinity