Úvod do teorie ortogonálních polynomů

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Metody algebry a funkcionální analýzy v aplikacích

Abstract

A brief survey of the theory of orthogonal polynomials is given including the fundamental recurrence relation and Favard's theorem, the Hamburger moment problem and its solution in terms of the Nevanlinna parametrization of probability measures in the indeterminate case, a relationship to the self-adjoint extensions of the associated Jacobi matrix and properties of the zeros of an orthogonal polynomial sequence, and the Gauss quadrature.

Furthermore, selected new results concerning a generalization of the Lommel polynomials are presented; a complete description can be found in the recent publication:

• F. Štampach, P. Šťovíček: *Orthogonal polynomials associated with Coulomb wave functions*,
J. Math. Anal. Appl. **419** (2014) 231-254
available online,
http://dx.doi.org/10.1016/j.jmaa.2014.04.049

The roots of the theory of orthogonal polynomials go back as far as to the end of the 18th century. The field of orthogonal polynomials was developed into considerable depths in the late 19th century from a study of continued fractions by P. L. Chebyshev and was further pursued by A. A. Markov and T. J. Stieltjes.

Some of the mathematicians who have worked on orthogonal polynomials include Gábor Szegő, Naum Akhiezer, Arthur Erdélyi, Wolfgang Hahn, Theodore Seio Chihara, Mourad Ismail, Waleed Al-Salam, and Richard Askey.

The theory of orthogonal polynomials is connected with many other branches of mathematics. Among others one can mention continued fractions, operator theory (Jacobi operators), moment problems, approximation theory and quadrature, stochastic processes (birth and death processes) and special functions.

Classical orthogonal polynomials

A scheme of classical orthogonal polynomials

- the Hermite polynomials
- the Laguerre polynomials, the generalized (associated) Laguerre polynomials
- the Jacobi polynomials, their special cases:
  - the Gegenbauer polynomials, particularly:
    - the Chebyshev polynomials
    - the Legendre polynomials

The interval of orthogonality

$$I = \begin{cases} \mathbb{R} & \text{for the Hermite polynomials} \\ (0, +\infty) & \text{for the generalized Laguerre polynomials} \\ (-1, 1) & \text{for the Jacobi (and Gegenbauer, \dots) polynomials} \end{cases}$$

Some common features:

$*$ Any classical orthogonal polynomial sequence, $\{\tilde{P}_n(x);\ n \geq 0\}$, forms an orthogonal basis in $\mathscr{H} = L^2(I, \varrho(x)\mathrm{d}x)$,
$I \subset \mathbb{R}$ is an open interval, $\varrho(x) > 0$ is continuous on I.

$*$ $\{\tilde{P}_n(x)\}$, after having been normalized to a sequence of *monic* polynomials, $\{P_n(x)\}$, obeys the recurrence relation

$$P_{n+1}(x) = (x - c_n)P_n(x) - d_n P_{n-1}(x), \quad n \geq 0,$$

$P_0(x) = 1$ and (conventionally) $P_{-1}(x) = 0$

c_n, $n \geq 0$, are all real, and d_n, $n \geq 1$, are all positive (d_0 is arbitrary)

$*$ The zeros of $P_n(x)$ are real and simple and belong all to I,
the zeros of $P_n(x)$ and $P_{n+1}(x)$ interlace,
the union of the zeros of $P_n(x)$ for all $n \geq 0$ is a dense subset in I

Charles Hermite: December 24, 1822 – January 14, 1901

- C. Hermite: *Sur un nouveau développement en série de fonctions*, (1864)
- P.L. Chebyshev: *Sur le développement des fonctions à une seule variable*, (1859)
- P. Laplace: *Mémoire sur les intégrales définies et leur application aux probabilités*, (1810)

Definition $(n = 0, 1, 2, \ldots)$

$$H_n(x) = n! \sum_{\ell=0}^{[n/2]} \frac{(-1)^\ell}{(n-2\ell)!\,\ell!}\,(2x)^{n-2\ell}$$

The Rodrigues formula

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\, e^{-x^2} = \left(2x - \frac{d}{dx}\right)^n \cdot 1$$

Orthogonality $\displaystyle\int_{-\infty}^{\infty} H_m(x) H_n(x)\, e^{-x^2}\, dx = 2^n n! \sqrt{\pi}\, \delta_{m,n}$,

an orthogonal basis of $\mathscr{H} = L^2(\mathbb{R}, e^{-x^2} dx)$

Recurrence relation

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \;\; n \geq 0; \; H_0(x) = 1, H_{-1}(x) = 0$$

Differential equation ($H_n(x)$ is a solution of Hermite's DE)

$$y'' - 2xy' + 2ny = 0$$

(Generalized) Laguerre polynomials

Edmond Laguerre: April 9, 1834 – August 14, 1886

• E. Laguerre: *Sur l'intégrale* $\int_x^\infty \frac{e^{-x}}{x}\,dx$, (1879)
• N. Y. Sonine: *Recherches sur les fonctions cylindriques et le développement des fonctions continues en séries*, (1880)

Definition $(L_n(x) \equiv L_n^{(0)}(x))$

$$L_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k!}\, x^k, \quad L_n^{(\alpha)}(x) = \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{x^k}{k!}$$

The Rodrigues formula

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha}e^x}{n!} \frac{d^n}{dx^n} \left(e^{-x} x^{n+\alpha}\right) = \frac{x^{-\alpha}}{n!} \left(\frac{d}{dx} - 1\right)^n x^{n+\alpha}$$

Orthogonality $(\alpha > -1)$

$$\int_0^\infty L_m^{(\alpha)}(x) L_n^{(\alpha)}(x)\, x^\alpha e^{-x}\, dx = \frac{\Gamma(n+\alpha+1)}{n!}\, \delta_{m,n}$$

an orthogonal basis of $\mathscr{H} = L^2((0,\infty), x^\alpha e^{-x} dx)$ $(\alpha > -1)$

Recurrence relation

$$(n+1) L_{n+1}^{(\alpha)}(x) = (2n+\alpha+1-x)\, L_n^{(\alpha)}(x) - (n+\alpha)\, L_{n-1}^{(\alpha)}(x), \quad n \geq 0,$$

$L_0^{(\alpha)}(x) = 1$ and, by convention, $L_{-1}^{(\alpha)}(x) = 0$

Differential equation ($L_n(x)$ is a solution of Laguerre's DE)

$$x\, y'' + (1-x)\, y' + n\, y = 0$$

Carl Gustav Jacob Jacobi: December 10, 1804 – February 18, 1851

- C.G.J. Jacobi: *Untersuchungen über die Differentialgleichung der hypergeometrischen Reihe*, J. Reine Angew. Math. **56** (1859) 149-165

$$P_n^{(\alpha,\beta)}(z) = \frac{\Gamma(\alpha+n+1)}{n!\,\Gamma(\alpha+\beta+n+1)} \sum_{m=0}^{n} \binom{n}{m} \frac{\Gamma(\alpha+\beta+n+m+1)}{\Gamma(\alpha+m+1)} \left(\frac{z-1}{2}\right)^m$$

The Rodrigues formula

$$P_n^{(\alpha,\beta)}(z) = \frac{(-1)^n}{2^n n!}\,(1-z)^{-\alpha}(1+z)^{-\beta}\,\frac{d^n}{dz^n}\left[(1-z)^{\alpha}(1+z)^{\beta}(1-z^2)^n\right]$$

Orthogonality $(\alpha, \beta > -1)$

$$\int_{-1}^{1} P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x)\,(1-x)^{\alpha}(1+x)^{\beta}\,dx = 0, \quad m \neq n$$

Recurrence relation $\;(P_0^{(\alpha,\beta)}(z) = 1,\ P_{-1}^{(\alpha,\beta)}(z) = 0)$

$$\begin{aligned} &2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)\,P_{n+1}^{(\alpha,\beta)}(z) \\ &= (2n+\alpha+\beta+1)\Big((2n+\alpha+\beta+2)(2n+\alpha+\beta)\,z + \alpha^2 - \beta^2\Big) P_n^{(\alpha,\beta)}(z) \\ &\quad - 2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)\,P_{n-1}^{(\alpha,\beta)}(z) \end{aligned}$$

Differential equation

$$(1-x^2)y'' + (\beta - \alpha - (\alpha+\beta+2)\,x)y' + n(n+\alpha+\beta+1)y = 0$$

Gegenbauer (ultraspherical) polynomials

Leopold Bernhard Gegenbauer: February 2, 1849 – June 3, 1903

- L. Gegenbauer: *Über einige bestimmte Integrale*, Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften. Mathematische-Naturwissenschaftliche Classe. Wien **70** (1875)
- L. Gegenbauer: *Über einige bestimmte Integrale*, (1876)
- L. Gegenbauer: *Über die Functionen* $C_n^\nu(x)$, (1877)

$$C_n^{(\alpha)}(z) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \, \frac{\Gamma(n-k+\alpha)}{\Gamma(\alpha)k!(n-2k)!} \, (2z)^{n-2k}$$

a particular case of the Jacobi polynomials

$$C_n^{(\alpha)}(z) = \frac{\Gamma(\alpha+1/2)\Gamma(2\alpha+n)}{\Gamma(2\alpha)\Gamma(n+\alpha+1/2)} \, P_n^{(\alpha-1/2,\alpha-1/2)}(z)$$

The Rodrigues formula

$$C_n^{(\alpha)}(z) = \frac{(-2)^n}{n!} \frac{\Gamma(n+\alpha)\Gamma(n+2\alpha)}{\Gamma(\alpha)\Gamma(2n+2\alpha)} (1-x^2)^{-\alpha+1/2} \frac{d^n}{dx^n} \left[(1-x^2)^{n+\alpha-1/2} \right]$$

Orthogonality $(\alpha, \beta > -1)$

$$\int_{-1}^{1} C_m^{(\alpha)}(x) C_n^{(\alpha)}(x) \, (1-x^2)^{\alpha-1/2} \, \mathrm{d}x = 0, \quad m \neq n$$

Recurrence relation $(C_0^{(\alpha)}(x) = 1, \, C_{-1}^{(\alpha)}(x) = 0)$

$$(n+1)C_{n+1}^{(\alpha)}(x) = 2x(n+\alpha)C_n^{(\alpha)}(x) - (n+2\alpha-1)C_{n-1}^{(\alpha)}(x), \quad n \geq 0$$

Differential equation (the Gegenbauer differential equation)

$$(1-x^2)y'' - (2\alpha+1)xy' + n(n+2\alpha)y = 0$$

Chebyshev polynomials of the first and second kind

Transliterations: Tchebycheff, Tchebyshev, Tschebyschow

Pafnuty Lvovich Chebyshev: May 16, 1821 – December 8, 1894

• P. L. Chebyshev: *Théorie des mécanismes connus sous le nom de parallélogrammes*, Mémoires des Savants étrangers présentés à l'Académie de Saint-Pétersbourg **7** (1854) 539–586

$T_0(x) = 1$, $U_0(x) = 1$, and for $n > 0$,

$$T_n(x) = \frac{n}{2}\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{n-k}\binom{n-k}{k}(2x)^{n-2k}$$

$$U_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k}(2x)^{n-2k}$$

Moreover, for all $n \geq 0$,

$$T_n(\cos(\vartheta)) = \cos(n\vartheta), \quad U_n(\cos(\vartheta)) = \frac{\sin((n+1)\vartheta)}{\sin\vartheta}$$

a particular case of the Gegenbauer polynomials

$$T_n(x) = \frac{n}{2\alpha}\, C_n^{(\alpha)}(x)\Big|_{\alpha=0} \ \ (\text{for } n \geq 1), \quad U_n(x) = C_n^{(1)}(x)$$

Orthogonality $(m \neq n)$

$$\int_{-1}^{1} T_m(x)T_n(x)\,\frac{dx}{\sqrt{1-x^2}} = 0, \quad \int_{-1}^{1} U_m(x)U_n(x)\sqrt{1-x^2}\,dx = 0$$

The Chebyshev polynomials $\{T_n(x)\}$ form an orthogonal basis of $\mathscr{H} = L^2((-1,1),(1-x^2)^{-1/2}\mathrm{d}x)$,

The Chebyshev polynomials $\{U_n(x)\}$ form an orthogonal basis of $\mathscr{H} = L^2((-1,1),(1-x^2)^{1/2}\mathrm{d}x)$

Recurrence relation

$$T_{n+1}(x) = (2-\delta_{n,0})\,xT_n(x) - T_{n-1}(x), \;\; U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$$

$T_0(x) = 1$, $U_0(x) = 1$ and, by convention, $T_{-1}(x) = 0$, $U_{-1}(x) = 0$

Differential equation

The Chebyshev polynomial $T_n(x)$ is a solution of the Chebyshev differential equation

$$(1-x^2)\,y'' - x\,y' + n^2\,y = 0,$$

the Chebyshev polynomial $U_n(x)$ is a solution of the differential equation

$$(1-x^2)\,y'' - 3x\,y' + n(n+2)\,y = 0$$

Legendre polynomials

Adrien-Marie Legendre: September 19, 1752 – January 10, 1833

• M. Le Gendre: *Recherches sur l'attraction des sphéroïdes homogènes*, Mémoires de Mathématiques et de Physique, présentés à l'Académie Royale des Sciences, par divers savans, et lus dans ses Assemblées **10** (1785) 411-435

Definition

$$P_n(x) = 2^n \sum_{k=0}^{n} \binom{n}{k} \binom{(n+k-1)/2}{n} x^k$$

a particular case of the Gegenbauer polynomials

$$P_n(x) = C_n^{(1/2)}(x)$$

The Rodrigues formula

$$P_n(x) = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} (x^2 - 1)^n$$

Orthogonality

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0 \quad \textit{for } m \neq n$$

an orthogonal basis of $\mathscr{H} = L^2((-1,1), \mathrm{d}x)$

Recurrence relation $(P_0(x) = 1,\ P_{-1}(x) = 0)$

$$(n+1) P_{n+1}(x) = (2n+1) x P_n(x) - n P_{n-1}(x), \quad n \geq 0$$

Differential equation (Legendre's differential equation)

$$\left((1 - x^2)\, y'\right)' + n(n+1) y = 0$$

Basic monographs

• G. Szegő: *Orthogonal Polynomials*, AMS Colloquium Publications, vol. XXIII, 2nd ed, (AMS, Rhode Island, 1958) [first edition 1939]

• J. A. Shohat, J. D. Tamarkin: *The Problem of Moments*, Math. Surveys, no. I, 2nd ed., (AMS, New York, 1950) [first edition 1943]

• N. I. Akhiezer: *The Classical Moment Problem and Some Related Questions in Analysis*, (Oliver & Boyd, Edinburgh, 1965)

• T. S. Chihara: *An Introduction to Orthogonal Polynomials*, (Gordon and Breach, Science Publishers, New York, 1978)

The moment functional

Definition

A linear functional $\mathcal{L}$ on $\mathbb{C}[x]$
(the linear space of complex polynomials in the variable x)
is called a *moment functional*, the number

$$\mu_n = \mathfrak{L}[x^n], \quad n = 0, 1, 2, \ldots,$$

is called a *moment of order n*.

Any sequence of moments $\{\mu_n\}$ determines unambiguously a moment functional $\mathfrak{L}$.

Definition

A moment functional $\mathfrak{L}$ is called *positive-definite*,
if $\mathcal{L}[\pi(x)] > 0$ for every polynomial $\pi(x)$ that is not identically zero and is non-negative for all real x.

Theorem

A moment functional $\mathfrak{L}$ *is positive-definite if and only if its moments* $\mu_n \in \mathbb{R}$

$$\Delta_n := \det(\mu_{j+k})_{j,k=0}^{n} = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix} > 0\,, \quad \forall n \geq 0$$

A real sequence $\{\mu_n;\ n \geq 0\}$ such that $\Delta_n > 0$, $\forall n \geq 0$, is said to be *positive*.

Definition

Given a positive-definite moment functional $\mathfrak{L}$, a sequence $\{\hat{P}_n(x);\ n \geq 0\}$ is called an *orthonormal polynomial sequence* with respect to $\mathfrak{L}$ provided for all $m, n \in \mathbb{Z}_+$,

1. $\hat{P}_n(x)$ is a polynomial of degree n
2. $\mathfrak{L}[\hat{P}_m(x)\hat{P}_n(x)] = \delta_{m,n}$

Quite frequently, it is convenient to work with a sequence of orthogonal monic polynomials, which we shall denote $\{P_n(x)\}$, rather than with the orthonormal polynomial sequence $\{\hat{P}_n(x)\}$.

Theorem

For every positive-definite moment functional $\mathfrak{L}$ there exists a unique monic orthogonal polynomial sequence $\{P_n(x)\}$.

Remark

$$\mathfrak{L}[P_n(x)^2] = \frac{\Delta_n}{\Delta_{n-1}}\,, \quad \forall n \geq 0 \ \ (\Delta_{-1} := 1)$$

hence

$$\hat{P}_n(x) = \sqrt{\frac{\Delta_{n-1}}{\Delta_n}}\, P_n(x)$$

are normalized,

$$P_n(x) = \frac{1}{\Delta_{n-1}} \begin{vmatrix} \mu_0 & \mu_1 & \dots & \mu_n \\ \mu_1 & \mu_2 & \dots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \dots & \mu_{2n-1} \\ 1 & x & \dots & x^n \end{vmatrix}$$

Let $\mathfrak{L}$ be a positive-definite moment functional,
let $\{\hat{P}_n(x)\}$ be the corresponding ON polynomial sequence.

$$\forall n \in \mathbb{N}, \forall \pi(x) \in \mathbb{C}[x], \deg \pi(x) < n \implies \mathfrak{L}[\hat{P}_n(x)\pi(x)] = 0$$

For $n = 0, 1, 2, \ldots,$

$$x\hat{P}_n(x) = \sum_{k=0}^{n+1} a_{n,k}\hat{P}_k(x), \quad a_{n,k} = \mathfrak{L}[x\hat{P}_n(x)\hat{P}_k(x)] \quad (a_{n,n+1} \neq 0)$$

For $k < n - 1$, $a_{n,k} = \mathfrak{L}[\hat{P}_n(x)\big(x\hat{P}_k(x)\big)] = 0$. Put

$$\alpha_n = \mathfrak{L}[x\hat{P}_n(x)\hat{P}_{n+1}(x)],\ \beta_n = \mathfrak{L}[x\hat{P}_n(x)^2].$$

α_n and β_n are all real and $\alpha_n = a_{n,n+1} \neq 0$.
$\{\hat{P}_n(x)\}$ fulfills the second-order difference relation

$$x\hat{P}_n(x) = \alpha_{n-1}\hat{P}_{n-1}(x) + \beta_n\hat{P}_n(x) + \alpha_n\hat{P}_{n+1}(x), \quad n \geq 0$$

$\hat{P}_0(x) = 1$, and we put $\hat{P}_{-1}(x) = 0$ (α_{-1} plays no role)

Rephrased: $c_n = \beta_n,\ d_n = \alpha_{n-1}^2$ (d_0 may be arbitrary)

Theorem

Let $\mathfrak{L}$ be a positive-definite moment functional, let $\{P_n(x)\}$ be the corresponding monic OG polynomial sequence. Then there exist real c_n, $n \geq 0$, and positive d_n, $n \geq 1$, such that

$$P_{n+1}(x) = (x - c_n)P_n(x) - d_n P_{n-1}(x), \quad n \geq 0,$$

with $P_0(x) = 1$ and $P_{-1}(x) = 0$.

Any $\mathcal{L}$ can be renormalized so that $\mathfrak{L}[1] = 1$

Theorem (Favard's Theorem)

Let c_n, $n \geq 0$, and d_n, $n \geq 1$, be arbitrary real and positive, respectively, let $\{P_n(x);\ n \in \mathbb{Z}_+\}$ be defined by the formula

$$P_{n+1}(x) = (x - c_n)P_n(x) - d_n P_{n-1}(x),\ \forall n \geq 0;\ P_{-1}(x) = 0,\ P_0(x) = 1$$

Then there exists a unique positive-definite $\mathfrak{L}$ such that

$$\mathfrak{L}[1] = 1, \quad \mathfrak{L}[P_m(x)P_n(x)] = 0 \quad \text{for } m \neq n,\ m, n = 0, 1, 2, \ldots$$

The zeros of an orthogonal polynomial sequence

Definition

Let $\mathcal{L}$ be a positive-definite moment functional and $E \subset \mathbb{R}$.
E is called a *supporting set* for $\mathcal{L}$ if $\mathcal{L}[\pi(x)] > 0$
for every real polynomial $\pi(x)$ which is non-negative on E
and does not vanish identically on E.

Theorem

Let $\mathcal{L}$ be a positive-definite moment functional, $\{P_n(x);\ n \geq 0\}$ be the corresponding monic orthogonal polynomial sequence.

For any n, the zeros of $P_n(x)$ are all real and simple, the zeros of $P_n(x)$ and $P_{n+1}(x)$ interlace, i.e. between any two subsequent zeros of $P_{n+1}(x)$ there is exactly one zero of $P_n(x)$.

On the contrary, if $2 \leq m < n$ then between any two zeros of $P_m(x)$ there is at least one zero of $P_n(x)$.

Moreover, if an interval I is a supporting set of $\mathcal{L}$ then the zeros of $P_n(x)$ are all located in the interior of I.

The Hamburger moment problem

Let $\{\mu_n;\ n = 0, 1, 2, \ldots\}$ be a sequence of moments defining a positive-definite moment functional $\mathfrak{L}$.
One can assume $\mu_0 = 1$, i.e. $\mathfrak{L}[1] = 1$.

One may ask whether $\mathfrak{L}$ can be defined with the aid of a probability measure $\mathrm{d}\sigma(x)$ on $\mathbb{R}$ where $\sigma(x)$ is a probability distribution,

$$\mathfrak{L}[\pi(x)] = \int_{-\infty}^{+\infty} \pi(x)\,\mathrm{d}\sigma(x), \quad \forall \pi(x) \in \mathbb{C}[x]$$

This can be reduced to

$$\int_{-\infty}^{+\infty} x^n\,\mathrm{d}\sigma(x) = \mu_n, \quad n = 0, 1, 2, \ldots$$

This problem is called the *Hamburger moment problem*.
The answer is always affirmative, but $\mathrm{d}\sigma(x)$ need not be unique.

The moment problem is *determinate* if there exists a unique probability measure $\mathrm{d}\sigma(x)$, *indeterminate* in the opposite case.

Define a sequence of polynomials $\{Q_n(x)\}$ by the recurrence

$$xQ_n(x) = \alpha_{n-1}Q_{n-1}(x)+\beta_n Q_n(x)+\alpha_n Q_{n+1}(x),\, Q_0 = 0,\, Q_1(x) = 1/\alpha_0$$

$Q_n(x)$ is called a *polynomial of the second kind*,
$\hat{P}_n(x)$ is called a *polynomial of the first kind*.

Theorem

If for some $z \in \mathbb{C} \setminus \mathbb{R}$, $\displaystyle\sum_{n=0}^{\infty} |\hat{P}_n(z)|^2 = \infty$, *then the Hamburger moment problem is determinate. Conversely, this equality holds true* $\forall z \in \mathbb{C} \setminus \mathbb{R}$ *if the Hamburger moment problem is determinate.*

Theorem

If for some $z \in \mathbb{C}$, $\displaystyle\sum_{n=0}^{\infty} \left(|\hat{P}_n(z)|^2 + |Q_n(z)|^2 \right) < \infty$ *then the Hamburger moment problem is indeterminate. Conversely, this inequality is fulfilled* $\forall z \in \mathbb{C}$ *if the Hamburger moment problem is indeterminate.*

The Nevanlinna parametrization

How can one describe all solutions to the moment problem in indeterminate case?
These series define entire functions, the *Nevanlinna functions*:

$$A(z) = z\sum_{n=0}^{\infty} Q_n(0)Q_n(z), \qquad B(z) = -1 + z\sum_{n=0}^{\infty} Q_n(0)\hat{P}_n(z),$$

$$C(z) = 1 + z\sum_{n=0}^{\infty} \hat{P}_n(0)Q_n(z), \qquad D(z) = z\sum_{n=0}^{\infty} \hat{P}_n(0)\hat{P}_n(z).$$

It is known that $A(z)D(z) - B(z)C(z) = 1$

Definition

Pick functions $\phi(z)$ are holomorphic functions on the open complex halfplane $\mathrm{Im}\, z > 0$, with values in $\mathrm{Im}\, z \geq 0$, extended to $\mathbb{C} \setminus \mathbb{R}$ by the formula $\phi(z) = \overline{\phi(\overline{z})}$ for $\mathrm{Im}\, z < 0$.

The set of Pick functions will be denoted by $\mathcal{P}$.

$\mathcal{P}$ may be augmented by the constant function $\phi(z) = \infty$.

Theorem

The formula $\displaystyle\int_{\mathbb{R}} \frac{d\sigma(x)}{z-x} = \frac{A(z)\phi(z)-C(z)}{B(z)\phi(z)-D(z)}\,, \quad z\in\mathbb{C}\setminus\mathbb{R},$

establishes a one-to-one correspondence between functions $\phi(z)\in\mathcal{P}\cup\{\infty\}$ *and solutions* $\sigma=\sigma_\phi$ *to the moment problem.*

Theorem (M. Riesz)

Let σ_ϕ *be a solution to an indeterminate moment problem,* $\phi(z)\in\mathcal{P}\cup\{\infty\}$. *Then* $\{\hat{P}_n(x);\, n=0,1,2,\ldots\}$ *is an ON basis in* $L^2(\mathbb{R},d\sigma_\phi)$ *if and only if* $\phi(z)=t$ *is constant, with* $t\in\mathbb{R}\cup\{\infty\}$.

The solutions σ_t, $t\in\mathbb{R}\cup\{\infty\}$, are referred to as *N-extremal*.

Proposition

The N-extremal solutions σ_t, $t\in\mathbb{R}\cup\{\infty\}$, are all purely discrete,

the measure $\displaystyle = \sum_{x\in\mathfrak{Z}_t}\rho(x)\,\delta_x$ where $\mathfrak{Z}_t=\{x\in\mathbb{R};\, B(x)t-D(x)=0\}$

$\rho(x) := \left(\sum_{n=0}^\infty \hat{P}_n(x)^2\right)^{-1} = (B'(x)D(x)-B(x)D'(x))^{-1}$

The associated Jacobi matrix

Recall the recurrence for an ON polynomial sequence $\{\hat{P}_n(x)\}$

$$x\hat{P}_n(x) = \alpha_{n-1}\hat{P}_{n-1}(x) + \beta_n\hat{P}_n(x) + \alpha_n\hat{P}_{n+1}(x), \quad n \geq 0$$

Let M be an operator on $\mathbb{C}[x]$ acting via multiplication by x,

$$M\pi(x) = x\,\pi(x), \quad \forall\pi(x) \in \mathbb{C}[x]$$

The matrix of M with respect to $\{\hat{P}_n(x)\}$ is the Jacobi matrix

$$\mathcal{J} = \begin{pmatrix} \beta_0 & \alpha_0 & & & \\ \alpha_0 & \beta_1 & \alpha_1 & & \\ & \alpha_1 & \beta_2 & \alpha_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

$\mathcal{J}$ represents a linear operator on the vector space of all complex sequences. For every $z \in \mathbb{C}$,

$$(\hat{P}_0(z), \hat{P}_1(z), \hat{P}_2(z), \ldots)$$

is a formal eigenvector of $\mathcal{J}$, $\mathcal{J}f = zf$
(f is unambiguous up to a scalar multiplier)

Let $\mathscr{D}$ be the linear hull of the canonical basis in $\ell^2(\mathbb{Z}_+)$;
$\mathscr{D}$ is $\mathcal{J}$-invariant. Denote $\dot{J} := \mathcal{J}\big|_{\mathscr{D}}$.
$\dot{J}$ is a symmetric operator on $\ell^2(\mathbb{Z}_+)$, $J_{\min} := \overline{\dot{J}}$, $J_{\max} = \mathcal{J}\big|_{\mathrm{Dom}\, J_{\max}}$,

$$\mathrm{Dom}\, J_{\max} = \{f \in \ell^2(\mathbb{Z}_+);\ \mathcal{J}f \in \ell^2(\mathbb{Z}_+)\}$$

Clearly, $\dot{J} \subset J_{\max}$. Straightforwardly,

$$(\dot{J})^* = (J_{\min})^* = J_{\max},\ (J_{\max})^* = J_{\min}$$

Hence $J_{\max}$ is closed and $J_{\min} \subset J_{\max}$.
The deficiency indices of $J_{\min}$ are either $(0,0)$ or $(1,1)$.
The latter case – if and only if for some and hence any $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\sum_{n=0}^{\infty} |\hat{P}_n(z)|^2 < \infty$$

An alternative terminology: $\mathcal{J}$ is *limit point* if the sequence $\{\hat{P}_n(z)\}$ is not square summable for some and hence any $z \in \mathbb{C} \setminus \mathbb{R}$,
$\mathcal{J}$ is *limit circle* in the opposite case.
In other words, $\mathcal{J}$ is limit point $\iff$ $\dot{J}$ is essentially self-adjoint

Theorem

J_{min} is self-adjoint iff the moment problem is determinate.
In the indeterminate case, the s.a. extensions of J_{min} are in one-to-one correspondence with the N-extremal solutions of the moment problem.
If J_t is a s.a. extension of J_{min} then the corresponding N-extremal solution $\sigma = \sigma_t$ is

$$\sigma_t(x) = \langle e_0, E_t\big((-\infty, x\,]\big)\, e_0\rangle,$$

E_t is the spectral projection-valued measure for J_t, e_0 is the first vector of the canonical basis, $\operatorname{supp}\sigma_t = \operatorname{spec} J_t$.

In the indeterminate case, the resolvent of any s.a. extension of J_{min} is a Hilbert-Schmidt operator.

Theorem

Suppose the moment problem is indeterminate. The spectrum of any s.a. extension J_t of J_{min} is simple and discrete.
Two different s.a. extensions J_t have distinct spectra.
Every $x \in \mathbb{R}$ is an eigenvalue of exactly one s.a. extension J_t .

Continued fractions

Let $\{a_n\}, \{b_n\} \subset \mathbb{C}$. A *generalized infinite continued fraction*

$$f = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{b_4 + \ddots}}}}$$

also written as

$$f = \frac{a_1 \mid}{\mid b_1} + \frac{a_2 \mid}{\mid b_2} + \frac{a_3 \mid}{\mid b_3} + \cdots,$$

is understood here as a sequence of convergents

$$f_n = \frac{A_n}{B_n}, \quad n = 1, 2, 3, \ldots$$

A_n, B_n are given by the fundamental *Wallis recurrence formulas*

$$A_{n+1} = b_{n+1} A_n + a_{n+1} A_{n-1}, \quad B_{n+1} = b_{n+1} B_n + a_{n+1} B_{n-1},$$

with $A_{-1} = 1,\ A_0 = 0,\ B_{-1} = 0,\ B_0 = 1$

Definition

Let $\mathcal{L}$ be a positive-definite moment functional,

$$P_{n+1}(x) = (x - c_n)P_n(x) - d_n P_{n-1}(x),\ \ n \geq 0,\ P_{-1}(x) = 0, P_0(x) = 1,$$

be the fundamental recurrence relation.
The monic polynomial sequence $\{P_n^{(1)}(x)\}$ defined by

$$P_{n+1}^{(1)}(x) = (x - c_{n+1})P_n^{(1)}(x) - d_{n+1}P_{n-1}^{(1)}(x),\ \ n \geq 0,$$

$P_{-1}^{(1)}(x) = 0, P_0^{(1)}(x) = 1$, is the *associated polynomial sequence*.

Proposition

Let $\{c_n;\ n = 0, 1, 2, \ldots\}$ and $\{d_n;\ n = 1, 2, 3, \ldots\}$ be a real and positive, respectively, $\{P_n\}$ and $\{P_n^{(1)}\}$ be defined as above. Then the convergents of the continued fraction

$$f = \frac{1\ |}{|\ x - c_0} - \frac{d_1\ |}{|\ x - c_1} - \frac{d_2\ |}{|\ x - c_2} - \cdots$$

are
$$f_n = \frac{P_{n-1}^{(1)}(x)}{P_n(x)} = \frac{Q_n(x)}{\hat{P}_n(x)},\ \ n = 1, 2, 3, \ldots$$

Gauss quadrature

Theorem

Let $\mathcal{L}$ be a positive-definite moment functional,
$\{P_n(x)\}$ the corresponding monic OG polynomial sequence.
Denote by $x_{n1} < x_{n2} < \ldots < x_{nn}$ the zeros of $P_n(x)$, $n \in \mathbb{N}$.
Then $\forall n \in \mathbb{N}$ there exists a unique n-tuple of numbers A_{nk},
$1 \leq k \leq n$,
such that for every polynomial $\pi(x)$ of degree at most $2n-1$,

$$\mathcal{L}[\pi(x)] = \sum_{k=1}^{n} A_{nk}\, \pi(x_{nk}).$$

The numbers A_{nk} are all positive.

Remark

Let $\{P_n^{(1)}\}$ designate the associated monic polynomial sequence. Then for $n, k \in \mathbb{N}$, $k \leq n$,

$$A_{nk} = \frac{P_{n-1}^{(1)}(x_{nk})}{P_n'(x_{nk})} = \left(\sum_{j=0}^{n-1} \hat{P}_j(x_{nk})^2 \right)^{-1}.$$

One also has

$$A_{nk} = \mathcal{L}[l_{nk}(x)^2]$$

where

$$l_{nk}(x) = \frac{P_n(x)}{(x - x_{nk})\, P_n'(x_{nk})}\,.$$

The Lommel polynomials

A fundamental property of Bessel functions is the recurrence

$$J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x) - J_{\nu-1}(x)$$

This relation can be iterated [Lommel, 1871] yielding,
for $n \in \mathbb{Z}_+$, $\nu \in \mathbb{C}$, $-\nu \notin \mathbb{Z}_+$ and $x \in \mathbb{C} \setminus \{0\}$,

$$J_{\nu+n}(x) = R_{n,\nu}(x) J_\nu(x) - R_{n-1,\nu+1}(x) J_{\nu-1}(x)$$

where

$$R_{n,\nu}(x) = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} \frac{\Gamma(\nu+n-k)}{\Gamma(\nu+k)} \left(\frac{2}{x}\right)^{n-2k}$$

is the so called *Lommel polynomial*.
Note that $R_{n,\nu}(x)$ is a polynomial in x^{-1} rather than in x.

- E. von Lommel: Mathematische Annalen **4** (1871) 103-116

The Lommel polynomials are directly related to Bessel functions,

$$R_{n,\nu}(x) = \frac{\pi x}{2} \left(Y_{-1+\nu}(x) J_{n+\nu}(x) - J_{-1+\nu}(x) Y_{n+\nu}(x) \right)$$

The Lommel polynomials obey the recurrence

$$R_{n+1,\nu}(x) = \frac{2\,(n+\nu)}{x}\,R_{n,\nu}(x) - R_{n-1,\nu}(x), \quad n \in \mathbb{Z}_+,$$

with the initial conditions $R_{-1,\nu}(x) = 0$, $R_{0,\nu}(x) = 1$.

The support of the measure of orthogonality for $\{R_{n,\nu+1}(x);\ n \geq 0\}$ coincides with the zero set of $J_\nu(z)$;

$j_{k,\nu}$ designates the k-th positive zero of $J_\nu(x)$, put $j_{-k,\nu} = -j_{k,\nu}$, $k \in \mathbb{N}$. The orthogonality relation reads

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{j_{k,\nu}^2}\,R_{n,\nu+1}(j_{k,\nu})R_{m,\nu+1}(j_{k,\nu}) = \frac{1}{2(n+\nu+1)}\,\delta_{m,n}$$

and is valid for all $\nu > -1$ and $m, n \in \mathbb{Z}_+$.

Let us also recall Hurwitz' limit formula

$$\lim_{n\to\infty} \frac{(\rho/2)^{\nu+n}}{\Gamma(\nu+n+1)}\,R_{n,\nu+1}(\rho) = J_\nu(\rho)$$

Lommel polynomials can also be addressed as polynomials in the parameter ν. The measure of orthogonality is supported on the zero set of a Bessel function regarded as a function of the order.

Let $\{T_n(u;\nu)\}_{n=0}^{\infty}$ be a sequence of polynomials in ν, depending on a parameter $u \neq 0$,

$$u\,T_{n-1}(u;\nu) - n\,T_n(u;\nu) + u\,T_{n+1}(u;\nu) = \nu T_n(u;\nu), \quad n \in \mathbb{Z}_+,$$

with $T_{-1}(u;\nu) = 0$, $T_0(u;\nu) = 1$. Then

$$T_n(u;\nu) = R_{n,\nu}(2u),\ \forall n \in \mathbb{Z}_+.$$

$J_\nu(x)$ as a function of ν, with $x > 0$, has infinitely many simple real zeros with no finite accumulation point.

Let $\theta_n = \theta_n(u)$, $n \in \mathbb{N}$, be the zeros of $J_{\nu-1}(2u)$ for $u > 0$, $\theta_n(-u) = \theta_n(u)$.

The corresponding Jacobi matrix $J(u;\nu)$ has the entries

$$\beta_n = -n,\ \alpha_n = u,\ n \in \mathbb{Z}_+.$$

It is an unbounded self-adjoint operator with a discrete spectrum.

The orthogonality measure for $\{T_n(u;\nu)\}$ is supported on the spectrum of $J(u;\nu)$,
the orthogonality relation has the form

$$\sum_{k=1}^{\infty} \frac{J_{\theta_k}(2u)}{u\left(\partial_z\big|_{z=\theta_k} J_{z-1}(2u)\right)} R_{n,\theta_k}(2u) R_{m,\theta_k}(2u) = \delta_{m,n}$$

Let us remark that initially this was Dickinson who formulated, in 1958, the problem of constructing the measure of orthogonality for the Lommel polynomials in the variable ν. Ten years later, Maki described such a construction.

• D. Dickinson: *On certain polynomials associated with orthogonal polynomials*, Boll. Un. Mat. Ital. **13** (1958) 116-124

• D. Maki: *On constructing distribution functions with application to Lommel polynomials and Bessel functions*, Trans. Amer. Math. Soc. **130** (1968), 281-297

Coulomb wave functions

Regular and irregular Coulomb wave functions, $F_L(\eta,\rho)$ and $G_L(\eta,\rho)$, are two linearly independent solutions to the ODE

$$\frac{\mathrm{d}^2u}{\mathrm{d}\rho^2}+\left(1-\frac{2\eta}{\rho}-\frac{L(L+1)}{\rho^2}\right)u=0$$

Wronskian formula

$$F_{L-1}(\eta,\rho)G_L(\eta,\rho)-F_L(\eta,\rho)G_{L-1}(\eta,\rho)=\frac{L}{\sqrt{L^2+\eta^2}}$$

$F_L(\eta,\rho)$ admits the decomposition

$$F_L(\eta,\rho)=C_L(\eta)\rho^{L+1}\phi_L(\eta,\rho)$$

where

$$C_L(\eta)=\frac{2^L\mathrm{e}^{-\pi\eta/2}\,|\Gamma(L+1+i\eta)|}{\Gamma(2L+2)}$$

and

$$\phi_L(\eta,\rho):=\mathrm{e}^{-i\rho}\,{}_1F_1(L+1-i\eta,2L+2,2i\rho)$$

The regular Coulomb wave function generalizes the Bessel function

$$
\begin{aligned}
F_{\nu-1/2}(0,\rho) &= \sqrt{\frac{\pi\rho}{2}}\,J_\nu(\rho) \\
\phi_{\nu-1/2}(0,\rho) &= \mathrm{e}^{-i\rho}\,{}_1F_1(\nu+1/2,2\nu+1,2i\rho) = \Gamma(\nu+1)\left(\frac{2}{\rho}\right)^{\nu} J_\nu(\rho)
\end{aligned}
$$

Moreover, it obeys the recurrence relation

$$
\begin{aligned}
&F_{L+1}(\eta,\rho) \\
&= \frac{L+1}{\sqrt{\eta^2+(L+1)^2}}\left(\left(\frac{1}{\rho}+\frac{\eta}{L(L+1)}\right)(2L+1)\,F_L(\eta,\rho)\right. \\
&\qquad\left. -\frac{\sqrt{\eta^2+L^2}}{L}\,F_{L-1}(\eta,\rho)\right)
\end{aligned}
$$

Orthogonal polynomials associated with $F_L(\eta,\rho)$

Put

$$w_N = \frac{\sqrt{\eta^2+(N+1)^2}}{(N+1)\sqrt{(2N+1)(2N+3)}} \quad \text{and} \quad \lambda_N = -\frac{\eta}{N(N+1)}\,,$$

for $\eta \in \mathbb{R}$, $N > -1/2$ and $N \neq 0$ if $\eta \neq 0$.

Let $\{P_n^{(L)}(\eta;z)\}_{n=0}^{\infty}$ be the sequence of OG polynomials given by

$$zP_n^{(L)}(\eta;z) = w_{L+n}P_{n-1}^{(L)}(\eta;z)+\lambda_{L+n+1}P_n^{(L)}(\eta;z)+w_{L+n+1}P_{n+1}^{(L)}(\eta;z),$$

with $P_{-1}^{(L)}(\eta;z)=0$, $P_0^{(L)}(\eta;z)=1$.

We restrict ourselves to the range of parameters

$-1 \neq L > -3/2$ if $\eta \in \mathbb{R}\setminus\{0\}$, and $L > -3/2$ if $\eta = 0$.

The associated Jacobi matrix J_L has the entries

$$\alpha_n^{(L)} = w_{L+n+1},\ \beta_n^{(L)} = \lambda_{L+n+1},\ \ n = 0,1,2,\ldots$$

Let us denote

$$R_n^{(L)}(\eta;\rho) := P_n^{(L)}(\eta;\rho^{-1})$$

for $\rho \neq 0$, $n \in \mathbb{Z}_+$.

Proposition

For $n \in \mathbb{Z}_+$ and $\rho \neq 0$ one has

$$R_n^{(L)}(\eta;\rho) = \frac{\sqrt{(L+1)^2+\eta^2}}{L+1}\sqrt{\frac{2L+2n+3}{2L+3}} \times \big(F_L(\eta,\rho)G_{L+n+1}(\eta,\rho) - F_{L+n+1}(\eta,\rho)G_L(\eta,\rho)\big).$$

Remark

The polynomials $R_n^{(L)}(\eta;\rho)$ are a generalization of the Lommel polynomials $R_{n,\nu}(x)$.

Indeed, the recurrence for $\{R_n^{(L)}(\eta;x)\}$ reduces to the recurrence for $\{R_{n,\nu}(x)\}$ if we let $\eta = 0$ and $L = \nu - 1/2$;

$$R_n^{(\nu-1/2)}(0;\rho) = \sqrt{\frac{\nu+n+1}{\nu+1}}\, R_{n,\nu+1}(\rho)$$

for $n \in \mathbb{Z}_+$, $\rho \in \mathbb{C} \setminus \{0\}$ and $\nu > -1$.
Similarly for other formulas to follow.

The polynomials $R_n^{(L)}(\eta,\rho)$ play the same role for Coulomb wave functions as Lommel polynomials do for Bessel functions.

Proposition

$$\begin{aligned} & R_n^{(L-1)}(\eta,\rho)F_L(\eta,\rho) \\ & -\frac{L+1}{L}\sqrt{\frac{2L+3}{2L+1}}\,\frac{\sqrt{\eta^2+L^2}}{\sqrt{\eta^2+(L+1)^2}}\,R_{n-1}^{(L)}(\eta,\rho)F_{L-1}(\eta,\rho) \\ & =\sqrt{\frac{2L+2n+1}{2L+1}}\,F_{L+n}(\eta,\rho), \end{aligned}$$

where $n\in\mathbb{Z}_+$, $0\neq L>-1/2$, $\eta\in\mathbb{R}$ and $\rho\neq 0$.

Proposition

For the above indicated range of parameters, the zeros of $\phi_L(\eta,\cdot)$ form a countable subset of $\mathbb{R}\setminus\{0\}$ with no finite accumulation points. The zeros of $\phi_L(\eta,.)$ are all simple, $\phi_L(\eta,\cdot)$ and $\phi_{L+1}(\eta,\cdot)$ have no common zeros, the zeros of the same sign of $\phi_L(\eta,\cdot)$ and $\phi_{L+1}(\eta,\cdot)$ mutually separate each other.

Let us arrange the zeros of $\phi_L(\eta,\cdot)$ into a sequence $\rho_{L,n}$, $n\in\mathbb{N}$ so that $0<|\rho_{L,1}|\leq|\rho_{L,2}|\leq|\rho_{L,3}|\leq\ldots$.

Theorem

For the above indicated range of parameters,

$$\sum_{k=1}^{\infty}\frac{1}{\rho_{L,k}^{2}}\,R_m^{(L)}(\eta;\rho_{L,k})R_n^{(L)}(\eta;\rho_{L,k})=\frac{(L+1)^2+\eta^2}{(2L+3)(L+1)^2}\,\delta_{m,n}$$

0 is not an eigenvalue of the compact operator J_L.
Here is a generalization of Hurwitz' limit formula

Proposition

For the above indicated range of parameters and $\rho\neq0$,

$$\lim_{n\to\infty}\sqrt{(2L+3)(2L+2n+1)}\,C_{L+n}(\eta)\rho^{L+n}\,R_{n-1}^{(L)}(\eta;\rho)$$
$$=\sqrt{1+\frac{\eta^2}{(L+1)^2}}\,F_L(\eta,\rho)$$