# Polynomial Solutions to Linear ODEs with Polynomial Coefs

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Telč, 19th August 2015

**Outline :**

1. Some Properties of Orthogonal Polynomials in 1D
2. Linear ODEs with Polynomial Coefficients
3. Non-degenerate Situations
4. Degenerate Squations
5. Bochner-Krall Systems, Recurrences, *etc.*

## *Some Properties of Orthogonal Polynomials in 1D:*

- polynomial solutions of hypergeometric type equation
- satisfy three-rerm recurrence relation and Christoffel-Darboux identities
- Rodrigues formula
- orthogonal $wrt$ positive measures (Favard's theorem)
- form bases of certain $L_2$-spaces
- quadrature formulae
- zeros are simple and interlacing (separation theorem)
- asymptotic distribution of zeros are known
- $\cdots$

*Equations of hypergeometric type:*

$$\sigma(s)y''(s) + \tau(s)y'(s) + \lambda y(s) = 0$$

$$deg(\sigma) = 2 \quad deg(\tau) = 1$$

reduced to the self-adjoint form

$$[\sigma(s)\varrho(s)y'(s)]' + \lambda\varrho(s)y(s) = 0$$

by choosing a function $\varrho$ such that $[\sigma(s)\varrho(s)]' = \tau(s)\varrho(s)$. The equation is usually considered on an interval $]a, b[$, chosen such that

$$\sigma(s) > 0, \varrho(s) > 0 \qquad \forall s \in ]a, b[$$
$$\lim_{s\to a} \sigma(s)\varrho(s) = \lim_{s\to b} \sigma(s)\varrho(s) = 0$$

Such an equation defines either a *finite* or an *infinite* system of orthogonal polynomials on $]a, b[$ depending on the set

$$\{\gamma \in \mathbb{R}| \lim_{s\to a} \sigma(s)\varrho(s)s^\gamma = \lim_{s\to b} \sigma(s)\varrho(s)s^\gamma = 0\}$$

We have a *unified* way to all these systems of orthogonal polynomials.

Let $\tau(s) = \tau_1 s + \tau_0$,

$$\text{for } \sigma(s) \in \{1, s, 1 - s^2\} \text{ we set } \Lambda = \infty$$

$$\text{for } \sigma(s) \in \{s^2 - 1, s^2, s^2 + 1\}, \text{ then } \Lambda = \frac{1-\tau_1}{2}$$

If $\lambda_n = -\frac{\sigma''(s)}{2} n(n-1) - \tau'(s)n$, then

$$\sigma(s)y_n''(s) + \tau(s)y_n'(s) + \lambda_n y_n(s) = 0,$$

where $y_n(s)$ is a $n$-degree polynomial satisfying Rodrigues formula

$$y_n(s) = \frac{B_n}{\varrho(s)} \frac{d^n}{ds^n}[\sigma^n(s)\varrho(s)].$$

$\{y_n^{\{\tau_1,\tau_0\}}(s)\}_{n<\Lambda}$ is a system of polynomials orthogonal with weight function $\varrho(s)$ on $]a, b[$.

$$\sqrt{\varrho(s)}y_n(s) \in L_2(a, b)$$

$$s y_n(s) = \alpha_n y_{n+1}(s) + \beta_n y_n(s) + \gamma_n y_{n-1} \text{ for } 1 < n+1 < \Lambda$$

The zeros of $y_n$ are simple and lie in the interval $]a, b[$ and the zeros of $y_n$ and $y_{n+1}$ interlace.

$\sigma(s) = s^2 - 1$

$$P_n^{((\tau_1-\tau_0)/2-1,(\tau_1+\tau_0)/2-1)}(-s)$$

$\sigma(s) = s^2$

$$\left(\frac{s}{\tau_0}\right)^n L_n^{1-\tau_1-2n}\left(\frac{\tau_0}{s}\right)$$

$\sigma(s) = s^2 + 1$

$$i^n P_n^{((\tau_1+i\tau_0)/2-1,(\tau_1-i\tau_0)/2-1)}(is)$$

S. Bochner (1929): Except for some trivial solutions of the form $y(x) = ax^n + bx^m$ and for some polynomials related to Bessel functions, the only solutions are Jacobi, Laguerre, and Hermite polynomials.

Bessel polynomials: $s^2\frac{d^2y}{ds^2} + (\tau_1 s + 2)\frac{dy}{ds} - n(n+\tau_1-1)y$

solution to linear ODE + orthogonality

It is desirable that the two important properties be valid at the same time. For higher degree linear ODE: Bochner-Krall problem

BKS is a system of real polynomials both orthogonal *w.r.t.* some inner product.

## *Heine–Stieltjes Problem*

Let us have a differential expression

$$T(z) = \sum_{i=1}^{k} Q_i(z) \frac{d^i}{dz^i},$$

$z \in \mathbb{C}$, $Q_i$'s are polynomials.

The number $r = \max_{i=1,\dots,k}(deg Q_i(z) - i)$ is called the *Fuchs index* of $T(z)$.

$T(z)$ with a negative Fuchs index can be easily transformed into the operator with a non-negative Fuchs index by the change of variable $z \to 1/z$.

**An Old Problem:** (E. Heine –T. Stieltjes)

Given a set of polynomials $Q_i$'s and a positive integer $n$ find all polynomials $V$ (Van Vleck polynomials) for which the equation

$$T(z)S(z) + V(z)S(z) = 0$$

has a polynomial solution $S$ (Stieltjes or Heine-Stieltjes polynomials) of degree $n$.

(Generally,) it is a multiparameter spectral problem.

The operator $T(z)$ is called *non-degenerate* if $deg Q_k(z) = k + r$.

*Existence of Polynomial Solutions (r=0):*

*i.e.* $degQ_i \leq i$, equality for at least one $i$.

**Theorem:** *For all sufficiently large integers $n$ there is a unique constant $\lambda_n$ and a monic polynomial $S_n$ of degree $n$ which satisfies*

$$\sum_{i=1}^{k} Q_i(z) \frac{d^i S_n(z)}{dz^i} = \lambda_n S_n(z).$$

*Moreover,* $\lambda_n = -\sum_{i=1}^{k} \binom{n}{i} \frac{d^i Q_i(z)}{dz^i}$.

**Theorem:** The polynomial spectrum $\{\lambda_n\}$ of $T(z)$ is simple for large $n$ and has the asymptotics

$$\lim_{n\to\infty} \frac{\lambda_n}{n(n-1)\dots(n-k+1)} = 1.$$

*Existence of Polynomial Solutions (general r):*

**Theorem:** For any non-degenerate $T(z)$ with algebraically independent coefficients of its polynomial coefficients $Q_i$ and for any $n \geq 0$ there exist exactly $\binom{n+r}{r}$ distinct Van Vleck polynomials $V(z)$'s whose corresponding Stieltjes polynomials $S(z)$'s are unique (up to a constant factor) and of degree $n$.

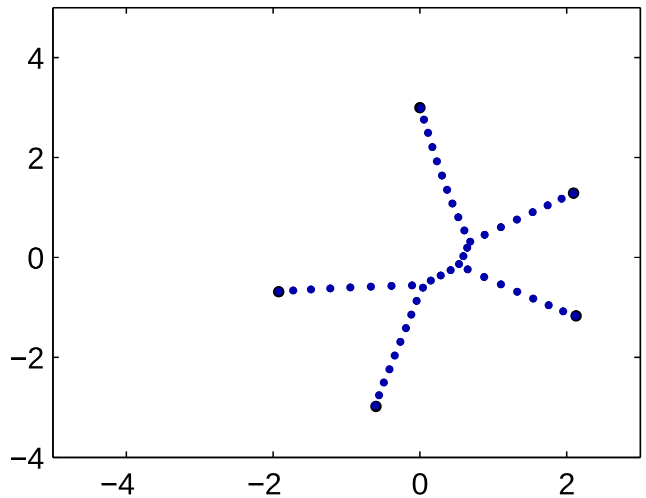
## *Asymptotic Distribution of Zeros (r=0):*

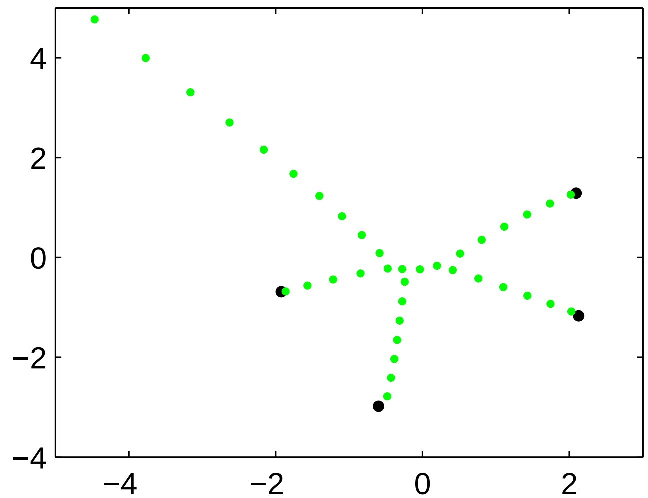
### An important topic

There is a difference between two cases:

- deg $Q_k = k$: *non–degenerate case*
- deg $Q_k < k$: *degenerate case*
- In the former case the union of all roots of all eigenpolynomials (when $n \to \infty$) is contained in a *compact* set (the closed convex hull of the set of all roots of $Q_k$).
- In the latter case the root of $S_n$ of the largest modulus tends to infinity when $n \to \infty$.

$Q_5(z)y^{(5)}(z) + Q_3(z)y^{(3)}(z) + \lambda y(z) = 0 \quad \tilde{Q}_5(z)y^{(5)}(z) + Q_3(z)y^{(3)}(z) + \lambda y(z) = 0$



*Non-degenerate case:*

*root-counting measure* $\mu_n = \frac{1}{n}\sum_{j=1}^{n} \delta(z - z_j)$, where $\delta(z-a)$ is the Dirac measure at $a$.

Then: (H. Rullgård)

**Theorem:** *Let $Q_k$ be monic. Then there exists a unique probability measure $\mu_{Q_k}$ with compact support whose Cauchy transform*

$$C(z) = \int \frac{d\mu_{Q_k}(\zeta)}{z - \zeta}$$

*satisfies $C(z)^k Q_k(z) = 1$ for almost all $z \in \mathbb{C}$.*

- *supp($\mu_{Q_k}$) is the union of finitely many smooth curve segments*
- *supp($\mu_{Q_k}$) contains all the zeros of $Q_k$*
- *supp($\mu_{Q_k}$) is contained in the convex hull of the zeros of $Q_k$*
- *supp($\mu_{Q_k}$) is connected and has connected complement*
- *Let $S_n$ be the monic eigenpolynomial of $T$ and $\mu_n$ the root-counting measure of $S_n$. Then $\mu_n$ converges weakly to $\mu_{Q_k}$ when $n \to \infty$. ($\mu_{Q_k}$ depends only on $Q_k$)*

$$\xi(z) = \int_a^z \frac{d\zeta}{\sqrt[k]{Q_k(\zeta)}}$$

a primitive function of $Q_k^{-1/k}$: $\xi$ locally defined function in any simply connected domain where $Q_k$ does not vanish

*(choice of branch and an integration constant is of no importance)*

- *the curve segments are mapped to straight lines by the mapping $\xi$*

Hence supp($\mu_{Q_k}$) can be thought of as a graph whose edges are smooth curve segments connecting certain vertices. The statement that supp($\mu_{Q_k}$) is connected and has connected complement then means that it is a connected graph without cycles, that is a tree.

*We know where; Question now is the distribution*

The measure $\mu$ can be reconstructed from the Cauchy transform

$$\mu = \frac{1}{\pi} \frac{\partial C_\mu(z)}{\partial \bar{z}}$$

**Arcsine theorem:** (G. Szegő) *If* $\{p_n(x)\}$ *is a system of polynomials orthogonal* w.r.t. *a weight* $\varrho(x)$ *supported on* $[-1,1]$ *such that* $\int_{-1}^{1} \ln \varrho(x) dx < \infty$ *then the asymptotic root-counting measure has the density*

$$\frac{1}{\pi\sqrt{1-x^2}}, \qquad x \in [-1,1].$$

Straightforward generalization:

Given two distinct roots $z_1 \neq z_2$ of $\sigma$ on $\mathbb{C}$, the *arcsine measure* $\omega_{[z_1,z_2]}$ is the measure supported on $[z_1, z_2]$ whose density at a point $t \in [z_1, z_2]$ equals

$$\frac{d\omega_{[z_1,z_2]}(t)}{dt} = \frac{1}{\pi\sqrt{|(t-z_1)(t-z_2)|}}.$$

If $\boxed{\deg \sigma < 2}$, then $T := \sigma(z)\frac{d^2}{dz^2} + \tau(z)\frac{d}{dz}$ is called **degenerate**.

Laguerre-type $\quad z\frac{d^2}{dz^2} + (\tau_1 z + \tau_0)\frac{d}{dz}$

Hermite-type $\quad \frac{d^2}{dz^2} + (\tau_1 z + \tau_0)\frac{d}{dz}$

**Proposition:** *The union of all roots of all eigenpolynomials is unbounded if and only if* $T$ *is degenerate. The Cauchy transform* $C(z)$ *of the asymptotic root measure* $\mu$ *of the scaled eigenpolynomial* $p_n(nz)$ *satisfies the algebraic equation* $zC(z) = 1$ *for almost all* $z$*.*

**Problem:** *Given a degenerate $T$, how fast does the root of maximal modulus $z_n$ grow?*

T. Bergkvist (2006):

$$\lim_{n\to\infty} \frac{|z_n|}{n^d} = c_T > 0 \qquad d = \max_{i\in[i_0+1,k]} \left(\frac{i-i_0}{i-degQ_i}\right),$$

where $i_0$ is the largest $i$ for which $deg(Q_i) = i$.

*The simplest situation:*

Let's study polynomial solutions to

$$y'' + (a_{1,1}z + a_{1,0})y' + \lambda y = 0,$$

where $a_{1,1}, a_{1,0} \in \mathbb{C}$ are parameters.

The solution can be written as

$$S_n(z) = z^n + \sum_{r=1}^{n} \binom{n}{r} \frac{1}{a_{1,1}^r} \sum_{j=0}^{r} (2j-1)!! \binom{r}{2j} a_{1,0}^{r-2j} a_{1,1}^j z^{n-r}$$

**Task:** Find the explicit dependence of the root of $S_n$ of maximal modulus $z_{max}$ on $n$.

## *Gräffe-Lobachevskii method:*

$$y(z) = z^n + \sum_{r=0}^{n-1} c_r z^r$$

The sum of the $\ell$-th powers of roots

$$s_\ell := \sum_{j=1}^{n} z_j^\ell$$

can be expressed as

$$s_\ell = -\ell c_\ell - \sum_{j=1}^{\ell-1} c_j s_{\ell-j}.$$

If there is a single root of maximal modulus we have

$$\lim_{\ell \to \infty} \frac{s_\ell}{z_{max}^\ell} = 1.$$

So, if we knew the general expression for $s_\ell$, we could find the exact growth of $z_{max}$.

Instead of numerics we can proceed analogically using *integer* or *rational* coefficients:

The sequence is formed by two subsequences (odd and even $s_\ell$). The odd one is

$$s_\ell = \frac{i(2i)^\ell \Gamma\left(\frac{\ell}{2}+1\right)}{\sqrt{\pi}\Gamma\left(\frac{\ell+3}{2}\right)} \frac{a_{1,0}\, n^\ell}{a_{1,1}^\ell} + \mathcal{O}(n^{\ell-1})$$

and the even one

$$s_\ell = \frac{(2i)^\ell \Gamma\left(\frac{\ell+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\ell}{2}+2\right)} \frac{n^\ell}{a_{1,1}^\ell} + \mathcal{O}(n^{\ell-1}).$$

Both subsequences have the same limit

$$\lim_{\ell\to\infty} \sqrt[\ell]{s_\ell} = 2\sqrt{-1}\sqrt{\frac{n}{a_{1,1}}}.$$

**N.B.:** The limit is complex and depends on $a_{1,1}$ only.

*General $k$:*

$$y^{(k)} - (a_{1,1}z + a_{1,0})y' + \lambda y = 0.$$

The solution can be written as

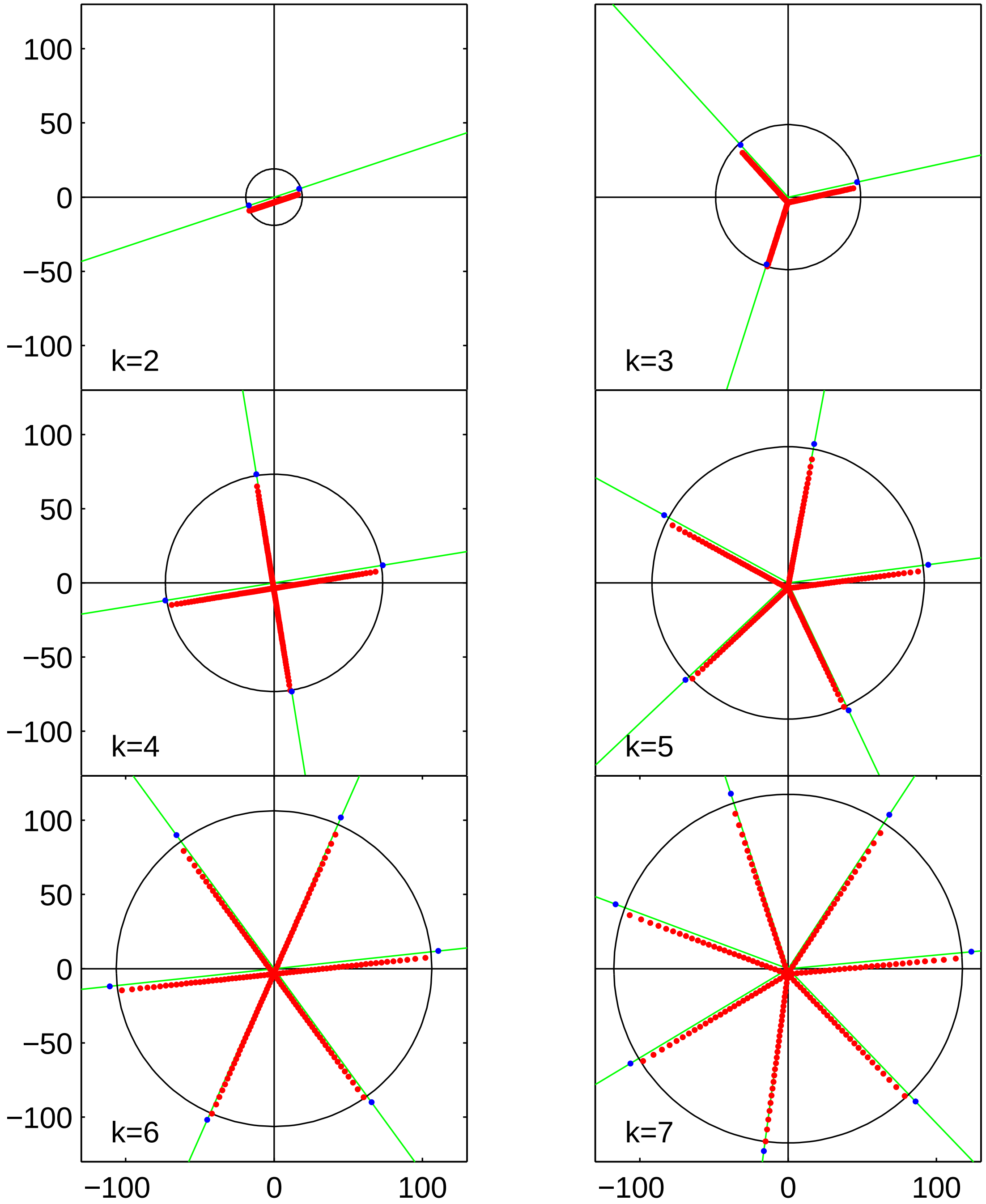
$$S_n^{(k)}(z) = z^n + \sum_{r=1}^{n} \binom{n}{r} \frac{1}{a_{1,1}^r} \left( a_{1,0}^r + \sum_{j=1}^{[r/k]} \binom{r}{kj} \frac{\Gamma(kj)}{k^{j-1}\Gamma(j)} \alpha_{1,1}^{(k-1)j} \alpha_{1,0}^{r-kj} \right) z^{n-r}$$

$$\lim_{\ell\to\infty} \sqrt[\ell]{s_\ell} = \frac{(-1)^{1/k} k}{(k-1)^{(k-1)/k}} \frac{\sqrt[k]{n^{k-1}}}{\sqrt[k]{\alpha_{1,1}}}$$

The sequence $s_\ell$ is formed by $k$ subsequences. The $k$-th root of $-1$ has $k$ values

$$\cos\left(\frac{(2\ell+1)\pi}{k}\right) + i\sin\left(\frac{(2\ell+1)\pi}{k}\right), \qquad \ell = 1, \ldots, k$$

and together with $1/\sqrt[k]{\alpha_{1,1}}$ ($\alpha_{1,1} \in \mathbb{C}$) it determines the direction (argument) of asymptotic line.



100
50
0
−50
−100
k=2
k=3
k=4
k=5
k=6
k=7
−100
0
100

*Next step:*

$$y^{(k)} + a_{2,2} z^2 y'' + a_{1,1} z y' + \lambda y = 0, \quad k > 2$$

$$S_n^{(k)}(z) = z^n + \sum_{r=1}^{[n/k]} \binom{n}{kr} \frac{\Gamma(kr)}{k^{r-1}\Gamma(r)} \frac{z^{n-kr}}{\prod_{j=1}^{r} (a_{1,1} + (2n - kj - 1) a_{2,2})}$$

$$\lim_{\ell \to \infty} \sqrt[\ell]{s_\ell} = \frac{(-1)^{1/k} \sqrt{k}}{2^{1/k} (k-2)^{(k-2)/(2k)}} \frac{\sqrt[k]{n^{k-2}}}{\sqrt[k]{\alpha_{2,2}}}.$$

general $k$ and $p$:

$$y^{(k)} + \sum_{r=1}^{p} a_{r,r} z^r y^{(r)} + \lambda y = 0, \quad k > p$$

$$S_n^{(k)}(z) = z^n + \sum_{r=1}^{[n/k]} \binom{n}{kr} \frac{\Gamma(kr)}{k^{r-1}\Gamma(r)} \frac{z^{n-kr}}{\prod_{j=1}^{r} \mathcal{A}_j^{(p)}}$$

$$\mathcal{A}_j^{(p)} = a_{1,1} + (2n - kj - 1) a_{2,2} + \left(3n^2 - 3(kj+2)n + (kj+1)(kj+2)\right) \alpha_{3,3} + \left(4n^3 - 6n^2(jk+3) + 2n\left(2j^2k^2 + 9jk + 11\right) - (jk+1)(jk+2)(jk+3)\right) \alpha_{4,4} + \ldots$$

$$\lim_{\ell \to \infty} \sqrt[\ell]{s_\ell} = \frac{(-1)^{1/k} \sqrt[p]{k}}{p^{1/k} (k-p)^{(k-p)/(pk)}} \frac{\sqrt[k]{n^{k-p}}}{\sqrt[k]{\alpha_{p,p}}}$$

*General $Q_k$:*

$$z^m y^{(k)}(z) + \alpha_{j,j} z^j y^{(j)}(z) + \alpha_{i,i} z^i y^{(i)}(z) + \lambda y(z) = 0,$$

where $m = 0, \ldots, k-1$, $k > j$, $i = 1, \ldots, j-1$

$$S_n^{(0,k,j,0)}(z) = z^n + \sum_{r=1}^{[n/k]} \binom{n}{kr} \frac{\Gamma(kr)}{k^{r-1}\Gamma(r)} \frac{z^{n-kr}}{\alpha_{j,j}^r \prod_{\ell=1}^r A_\ell^{(k,j)}(n)}$$

$$\begin{aligned}
A_\ell^{(k,1)}(n) &= 1 \\
A_\ell^{(k,2)}(n) &= 2n - (k\ell + 1) \\
A_\ell^{(k,3)}(n) &= 3n^2 - 3(k\ell + 2)n + (k\ell + 1)(k\ell + 2) \\
A_\ell^{(k,4)}(n) &= 4n^3 - 6(k\ell + 3)n^2 + 2(2k^2\ell^2 + 9k\ell + 11)n - (k\ell + 1)(k\ell + 2)(k\ell + 3) \\
A_\ell^{(k,5)}(n) &= 5n^4 - 10(k\ell + 4)n^3 + 5(2k^2\ell^2 + 12k\ell + 21)n^2 - \\
&\quad 5(k\ell + 4)(k^2\ell^2 + 4k\ell + 5)n + (k\ell + 1)(k\ell + 2)(k\ell + 3)(k\ell + 4) \\
A_\ell^{(k,6)}(n) &= 6n^5 - 15(k\ell + 5)n^4 + 10(2k^2\ell^2 + 15k\ell + 34)n^3 - \\
&\quad 15(k\ell + 5)(k^2\ell^2 + 5k\ell + 9)n^2 + \\
&\quad (6k^4\ell^4 + 75k^3\ell^3 + 340k^2\ell^2 + 675k\ell + 548)n - \\
&\quad (k\ell + 1)(k\ell + 2)(k\ell + 3)(k\ell + 4)(k\ell + 5) \\
&\quad \ldots
\end{aligned}$$

$$
\begin{aligned}
A_\ell^{(k,j)}(n) \quad = \quad & \binom{n}{1}\ \mathbf{n^{j-1}} - \\
& \binom{n}{2}\ (k\ell + j - 1)\mathbf{n^{j-2}} + \\
& \binom{n}{3}\ (k^2\ell^2 + 3(j-1)/2\, k\ell + (3j-1)(j-2)/4)\mathbf{n^{j-3}} - \\
& \binom{n}{4}\ (k\ell + j - 1)(k^2\ell^2 + (j-1)k\ell + j(j-3)/2)\mathbf{n^{j-4}} + \\
& \binom{n}{5}\ \Big(k^4\ell^4 + \tfrac{5}{2}(j-1)k^3\ell^3 + \tfrac{5}{6}(j-2)(3j-1)k^2\ell^2 + \\
& \qquad \tfrac{5}{4}(j-3)(j-1)j\, k\ell + \tfrac{1}{48}(j-4)\left(15j^3 - 30j^2 + 5j + 2\right)\Big)\ \mathbf{n^{j-5}} - \\
& \binom{n}{6}\ (k\ell + j - 1)\left(k^4\ell^4 + 2(j-1)k^3\ell^3 + \tfrac{1}{4}\left(7j^2 - 19j + 2\right)k^2\ell^2 + \right. \\
& \qquad \left. \tfrac{1}{4}(j-1)\left(3j^2 - 11j - 2\right)k\ell + \tfrac{1}{16}(j-5)j\left(3j^2 - 7j - 2\right)\right)\ \mathbf{n^{j-6}} + \\
& \qquad \cdots \\
+ \quad & (-1)^{j-1}\prod_{s=1}^{j-1}(k\ell + s)\mathbf{n^0}.
\end{aligned}
$$

$$z^m y^{(k)}(z) + \alpha_{j,j} z^j y^{(j)}(z) + \alpha_{i,i} z^i y^{(i)}(z) + \lambda y(z) = 0$$

$$S_n^{(m,k,j,i)}(z) = z^n + \sum_{r=1}^{[n/k]} \binom{n}{kr} \frac{\Gamma(kr)}{k^{r-1}\Gamma(r)} \frac{(m-k)^{m(r-1)} \prod_{\ell=1}^{m} \left(\frac{k-\ell-n}{k-m}\right)_{r-1} z^{n-kr}}{\prod_{\ell=1}^{r} \left(\alpha_{j,j} A_\ell^{(k-m,j)}(n) + \alpha_{i,i} A_\ell^{(k-m,i)}(n)\right)}$$

$$z_{max} \sim \frac{(-1)^{\frac{1}{k-m}} k^{\frac{k}{j(k-m)}}}{j^{\frac{1}{k-m}} (k-j)^{\frac{k-j}{j(k-m)}}} \frac{n^{\frac{k-j}{k-m}}}{\sqrt[k-m]{\alpha_{j,j}}}, \qquad n \to \infty.$$

*Conclusions of this part:*

1. This result confirms that the asymptotic behaviour of $z_{max}$ is determined by $m, k, j$, and $\alpha_{j,j}$, *i.e.* the lower terms in coefficient polynomials $Q_r, r = 1, \ldots, j-1$ do not influence it.

2. There is always at least one branch of roots that runs to infinity, (*i.e.* $|z_{n_i}| \to \infty$ as $i \to \infty$). The number and direction(s) of the(se) branch(es) is determined by $(-1/\alpha_{j,j})^{1/(k-m)}$, *i.e.* by the difference $k - m$ and $\arg(\alpha_{j,j})$.

## *Classical Heun Equation (r=1):*

The classical Heun equation has the form

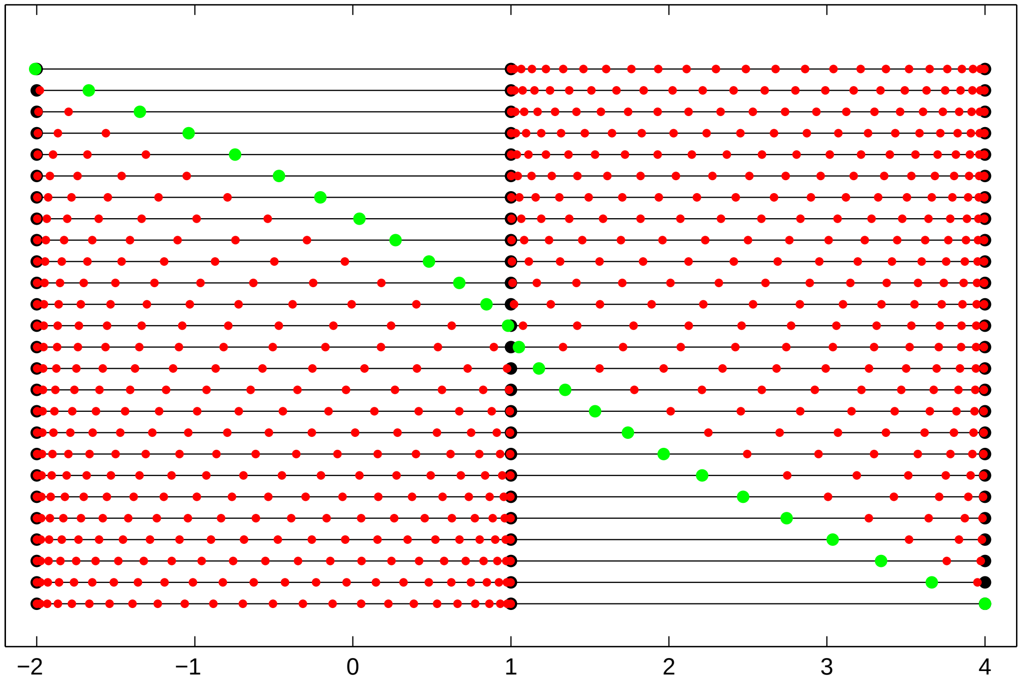
$$\left\{Q(z)\frac{d^2}{dz^2}+P(z)\frac{d}{dz}+V(z)\right\}S(z)=0,$$

$$deg(Q)=3 \quad deg(R)=2 \quad deg(V)=1$$

Some **answers** have been found:

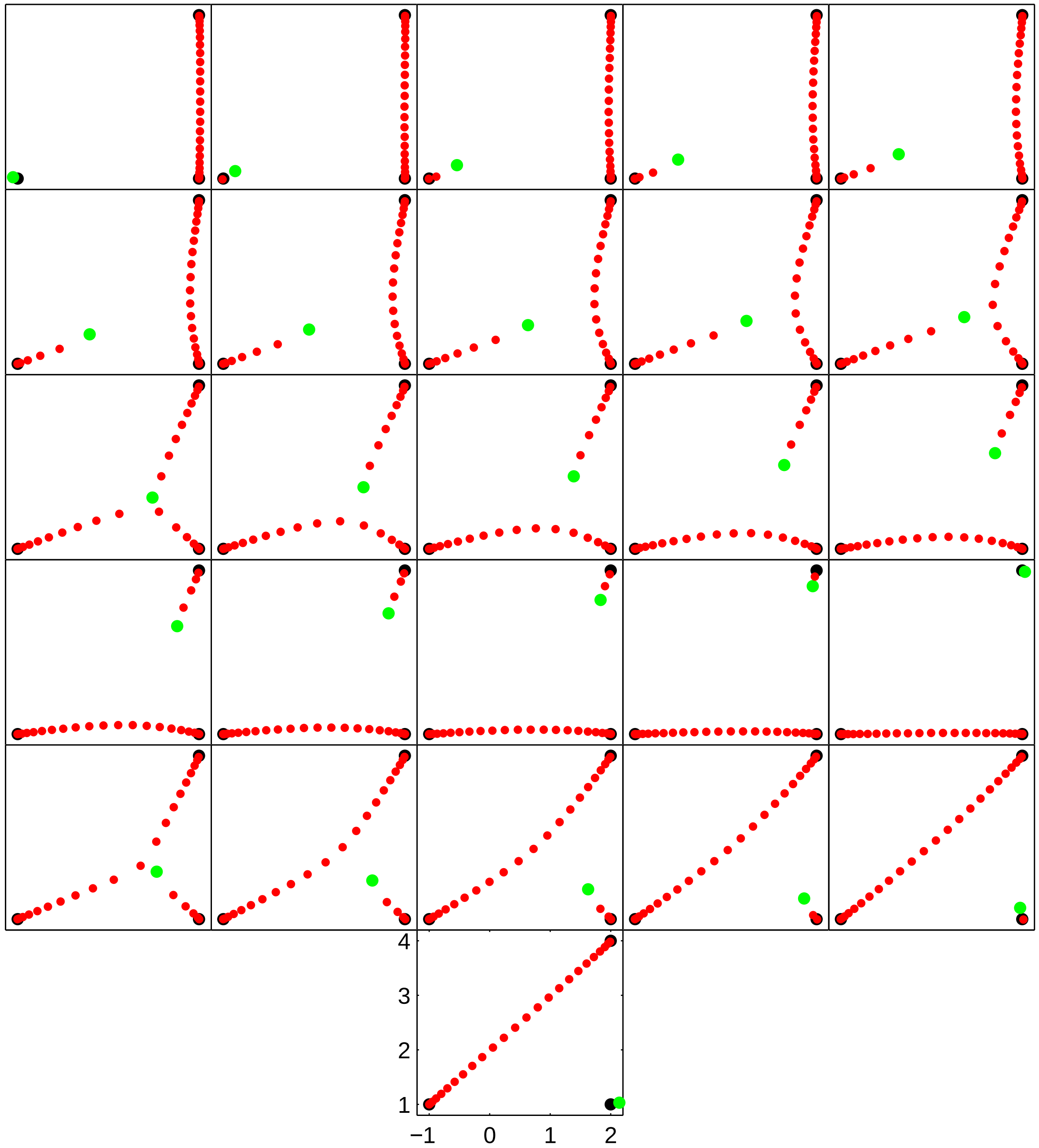
- For a generic pair $(Q,P)$ and any positive $n$ there exist $n+1$ distinct Van Vleck polynomials $V$. However, $n+1$ is an upper bound.
- In the case of the $classical\ Lamé$ equation, *i.e.* $P=Q'/2$, and if $Q$ has three distinct $real$ roots $a_1<a_2<a_3$, Stieltjes (1885) proved that the roots of any $V$ and $S$ are real and simple and belong to the interval $(a_1,a_3)$. None of the roots of $S$ coincides with any of $V$.

- $n+1$ polynomials $S$ are on 1-1–correspondence with $n+1$ ways of distributing $n$ points (roots) into open intervals $(a_1, a_2)$ and $(a_2, a_3)$:




- When the roots of $Q$ are *complex* and the rational function $\frac{P(z)}{Q(z)} = \sum_{i=0}^{2} \frac{\rho_i}{z-a_i}$ has all positive residues then any root of $V$ and any root of $S$ lies in $\Delta_Q$, the (closed) convex hull of the set of all roots of $Q$, *i.e.* $a_1, a_2, a_3$ (Pólya 1912). When the residues are negative the situation is different (both types of roots may lie outside of $\Delta_Q$).

Numerical examples suggest that the roots of Van Vleck polynomials sit on some curves contained in $\Delta_Q$ when $n \to \infty$. The same applies to roots of Stieltjes polynomials:



## *Root Localization of Spectral Polynomials:*

- For any $\epsilon > 0$ there exists $N_\epsilon$ such that for any $n \geq N_\epsilon$ any root of any $V$ having $S$ of degree $n$ as well as any root of this $S$ lie in the $\epsilon$-neighbourhood of $\Delta_Q$ (in the usual Euclidean distance on $\mathbb{C}$), *i.e.* it is independent of $P$.

It is convenient to introduce a sequence $\{Sp_n(\lambda)\}$ of *spectral polynomials* where the $n$th spectral polynomial is defined as

$$Sp_n(\lambda) = \prod_{j=1}^{n+1} (\lambda - z_{n,j}),$$

$z_{n,j}$ being the unique root of the $j$th Van Vleck polynomial (in any fixed ordering).

- If roots of $Q$ are real zeros of two successive spectral polynomials interlace. In spite of the fact that these polynomials have simple zeros that interlace, the system $\{Sp_n\}$ is not orthogonal with respect to any measure.

  *The proof is based on the finding that the asymptotic zero distribution of $Sp_n$ is different from that of orthogonal polynomials, showing also that $Sp_n$ do not obey any three-term recurrence relation.*

Let $a_1, a_2, a_3$ are roots of $Q(z)$. For $i \in \{1,2,3\}$ denote $\gamma_i$ the curve

$$\left\{ b \in \mathbb{C} \,\middle|\, \int_{a_j}^{a_k} \sqrt{\frac{b-t}{(t-a_1)(t-a_2)(t-a_3)}} dt \in \mathbb{R} \right\}.$$

These three curves connect $a_i$ with a unique common point $b_0$ in the interior of triangle determined by $a_1, a_2, a_3$. Denote the segment of $\gamma_i$ from $a_i$ to this point by $\Gamma_i$.

- *The sequence $\{\mu_n\}$ of the root-counting measures of spectral polynomials converges to a probability measure $\mu$ supported on $\Gamma_Q = \Gamma_1 \bigcup \Gamma_2 \bigcup \Gamma_3$, which depends only on $Q$.*
- *$C_\mu$ satisfies a linear non-homogeneous differential equation*

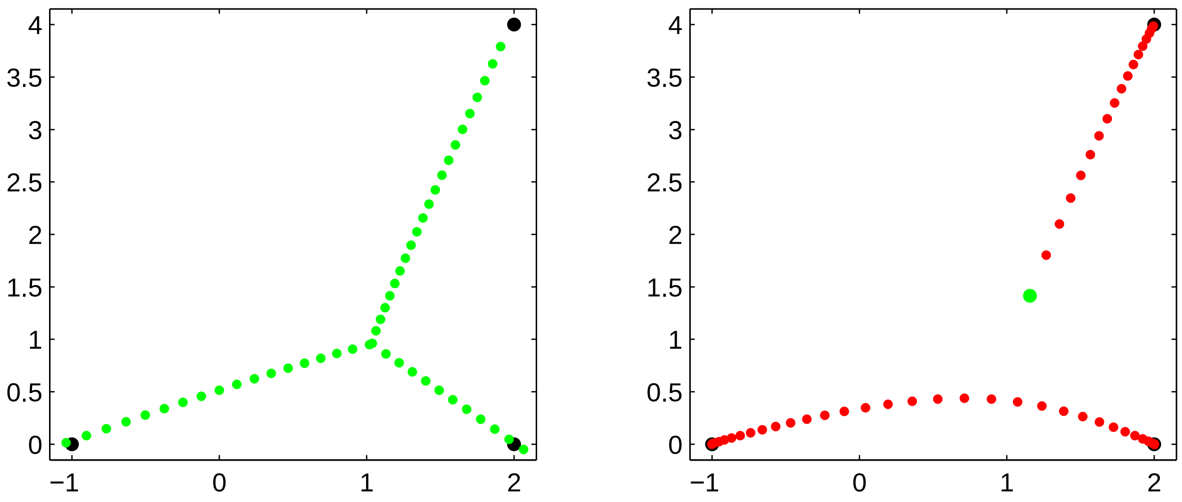
$$Q(z)C_\mu''(z) + Q'(z)C_\mu'(z) + Q''(z)C_\mu(z)/8 + Q'''(z)/24 = 0$$

  whose only singularities are $a_1, a_2, a_3$, and $\infty$. The unique solution with asymptotics $1/z$ near infinity can be extended to the whole $\mathbb{C} \setminus \Gamma_Q$ and $C_\mu$ coincides with this solution. But then the density of $\mu$ is the $\bar{z}$-derivative of this solution restricted to $\mathbb{C} \setminus \Gamma_Q$ and cannot vanish at any generic point of $\Gamma_Q$.

Given a finite (complex-valued) measure $\mu$ supported on $\mathbb{C}$, the *Cauchy transform* $C_\mu(z)$ is analytic outside the support of $\mu$ and has a number of important properties, among others

$$\mu = \frac{1}{\pi} \frac{\partial C_\mu(z)}{\partial \bar{z}} \qquad pot_\mu(z) = \int_{\mathbb{C}} \log|z-\zeta| d\mu(\zeta).$$

## *Root Localization of Stieltjes Polynomials:*




Take any sequence $\{S_{n,i_n}\}$, $deg(S_{n,i_n}) = n$ such that the sequence of monic Van Vleck polynomials $\{\widetilde{V}_{n,i_n}\}$ converges to some limiting $\widetilde{V}$ (root $\widetilde{b} \in \Gamma_Q$).

**Theorem:** *The sequence $\{\nu_{n,i_n}\}$ of the root-counting measures of Stieltjes polynomials $\{S_{n,j_n}\}$ converges weakly to a probability measure $\nu_{\widetilde{V}}$ whose Cauchy transform $C_{\widetilde{V}}$ satisfies*

$$C^2_{\widetilde{V}}(z) = \frac{\widetilde{V}(z)}{Q(z)}.$$

*What is the support of $\nu_{\widetilde{V}}$*

Denote $U_1$ and $U_2$ arbitrary monic complex polynomials for which $\deg U_2 - \deg U_1 = 2$ and denote

$$R(z) = -\frac{U_1(z)}{U_2(z)}.$$

Zeros and poles of $R$ constitute the set of *singular points* of $R$. Nonsingular points are called *regular*.

Trajectories of $R$ can be naturally parametrized by their arclength. In a neighbourhood of a regular point $z_0$ we can introduce a local *canonical* parameter by

$$w(z) := \int_{z_0}^{z} \sqrt{R(\xi)} d\xi.$$

So called *horizontal trajectories* in the $z$-plane correspond to horizontal straight lines in the $w$-plane, *i.e.* they are defined by $\Im w = const$.

A trajectory of $R$ is called *singular* if there exists a singular point of $R$ belonging to its closure.

A non-singular trajectory $\gamma_{z_0}(t)$ is called *closed* if there exists a $T > 0$ (a period) such that $\gamma_{z_0}(t+T) = \gamma_{z_0}(t)$ for all $t \in \mathbb{R}$.

If the set of all closed trajectories covers the Riemann surface up to a set of Lebesgue measure zero then the union of all singular trajectories and singular points of $R$ is compact.

Let $K_\infty$ is a compact union of all singular trajectories and singular points except $\infty$. Then $\mathbb{C}\backslash K_\infty$ is connected.

Let $f \in L^1_{loc}$ such that $f(z) \to 0$ as $z \to \infty$ and let $\nu$ be a compactly supported measure in $\mathbb{C}$ such that $\partial f/\partial \bar{z} = -\pi\nu$ in the sense of distributions. Then $f(z) = C_\nu$ *a.e.*

For $R$, there exists a real (*i.e.* signed) and compactly supported measure $\nu$ of total mass 1 (*i.e.* $\int_{\mathbb{C}} d\nu = 1$) whose Cauchy transform $C_\nu$ satisfies the equation

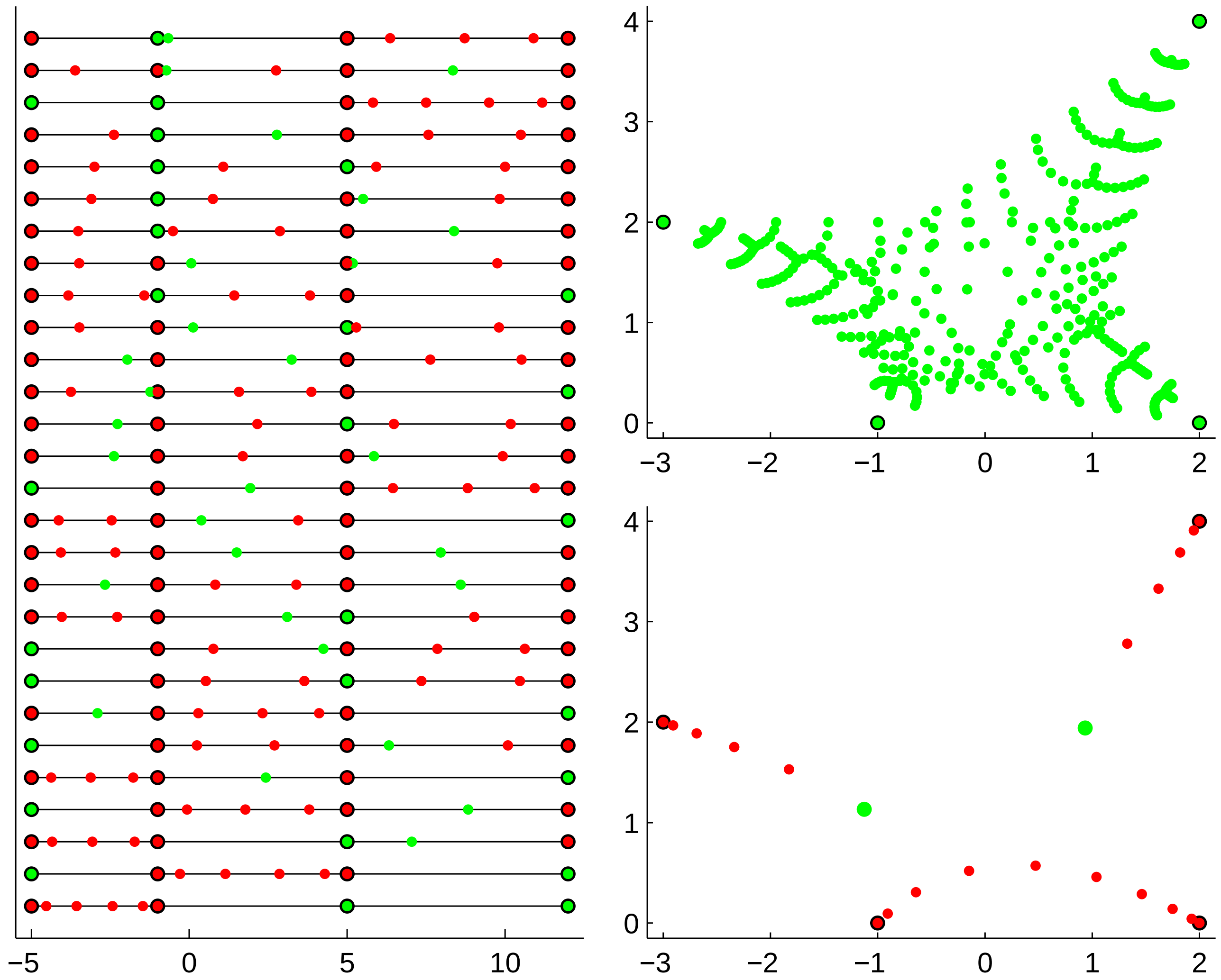
$$C_\nu^2(z) = \tfrac{U_1(z)}{U_2(z)}.$$

For $R$, there exist exactly $2^{d-1}$ real measures whose Cauchy transforms satisfy the above equation *a.e.* and whose support is contained in $K_R = K_\infty \bigcup \{\infty\}$. Here $d$ is the total number of connected components of the complement (including the infinite component, *i.e.* containing $\infty$).

For $R$, there exists at most one positive measure among those $2^{d-1}$ real measures mentioned above. Its support belongs necessarily to $K_R$.

*The support of $\nu_{\widetilde{V}}$ consists of (horizontal) singular trajectories of $R$.*

# *Higher Fuchs Indices (r=2):*



## *Bochner-Krall Systems:*

Polynomial systems $\{p_n\}_{n=0}^{\infty}$ orthogonal *wrt* classical Lebesgue-type bilinear forms of the type $\int pq d\mu$, where $\mu$ is a (possibly signed) real, finite Borel measure, *i.e.*

- $\int p_n p_m d\mu = k_n \delta_{n,m}$
- $T(x) p_n(x) = \lambda_n p_n(x)$

$$\{p_n\}_{n=0}^{\infty} \in \mathit{BKS}(N)$$

For a differential expression with real, sufficiently differentiable coefficients be Lagrangian symmetrizable, then it must be of even order $N$.

- H.L.Krall proved that BKS$(2N+1) = \emptyset$
- Bochner, Lesky determined BKS(2) under a complex linear change of variable
- Krall classified BKS(4) under a complex linear change of variable
- A complete determination of BKS(6) is still unknown

The most interesting class of OP eigenfunctions of ODEs is form the perspective of *spectral theory of self-adjoint differential operators*.

Surprisingly, Jung, Kwon, and Lee (1997) discovered a *not* Lagrangian symmetrizable expression (fourth-order differential expression) orthogonal *wrt* a (non-classical) positive-definite inner product. This expression generates a *self-adjoint* operator with the $\{p_n\}_{n=0}^{\infty} \notin \mathrm{BKS}(4)$ as eigenfunctions in some Hilbert-Sobolev space $\mathcal{H}$ and it is symmetric in the inner product of $\mathcal{H}$.

No longer does orthogonality mean classical inner product. New examples are emerging, *e.g. Sobolev orthogonal polynomials*; a generalization of the classification is needed.

*E.g.* Sobolev OPs need not satisfy a three-term recurrence relation, unlike their classical counterparts.