Construction of Representations and Lie Fields

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Outline

Universal Enveloping Algebras

Definition

Let $\mathbb{F}$ be a field. A **Lie algebra** over $\mathbb{F}$ is an $\mathbb{F}$-vector space $\mathfrak{g}$ equipped with a bilinear map $[\ ,\ ]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ fulfilling the two following conditions:

$$[x, x] = 0, \quad x \in \mathfrak{g}, \tag{1}$$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \quad x, y, z \in \mathfrak{g}. \tag{2}$$

Universal Enveloping Algebras

Definition

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{F}$. Choose a basis $(x_1, \dots, x_n)$ and let $a_{ij}^n \in \mathbb{F}$ denote the numbers satisfying

$$[x_i, x_j] = \sum_{k=1}^{n} a_{ij}^n x_k, \quad 1 \leq i, j \leq n. \tag{3}$$

Then the **universal enveloping algebra** $U(\mathfrak{g})$ of $\mathfrak{g}$ is defined to be the unital associative algebra over $\mathbb{F}$, generated by $X_1, \dots, X_n$, subject to the relations

$$X_i X_j - X_j X_i = \sum_{k=1}^{n} a_{ij}^n X_k, \quad 1 \leq i, j \leq n. \tag{4}$$

Lie Fields

- given an enveloping algebra $U(\mathfrak{g})$, consider the following equivalence relation on $U(\mathfrak{g})\backslash\{0\} \times U(\mathfrak{g})$:

$$(v_1, u_1) \sim (v_2, u_2) \iff \begin{array}{c} \text{there are } w, t \in U(\mathfrak{g})\backslash\{0\} \text{ s.t.} \\ wv_1 = tv_2 \text{ and } wu_1 = tu_2. \end{array} \tag{5}$$

- let $\frac{u_1}{v_1} \equiv [v_1, u_1] \subset U(\mathfrak{g})\backslash\{0\} \times U(\mathfrak{g})$ denote the class of equivalence, defined by (5), containing (v_1, u_1)

Lie Fields

Definition

The **Lie field** $D(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is the set of all the equivalence classes established above, together with addition and multiplication defined by

$$\frac{u_1}{v_1} + \frac{u_2}{v_2} := \frac{w_1 u_1 + w_2 u_2}{w_1 v_1}, \tag{6}$$

$$\frac{u_1}{v_1} \cdot \frac{u_2}{v_2} := \frac{t_1 u_2}{t_2 v_1}, \tag{7}$$

where $w_1, w_2, t_2 \in U(\mathfrak{g}) \backslash \{0\}$ and $t_1 \in U(\mathfrak{g})$ satisfy

$$w_1 v_1 = w_2 v_2 \quad \text{and} \quad t_2 u_1 = t_1 v_2. \tag{8}$$

Lie Fields

- each non-zero $\frac{a}{b} \in D(\mathfrak{g})$ has its inverse element: $\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$
- we may identify $\frac{u}{1} \equiv u$
- then for $0 \neq u \in U(\mathfrak{g})$ we denote $u^{-1} \equiv \frac{1}{u}$ and we may write
$$\frac{u}{v} = \frac{1}{v} \cdot \frac{u}{1} = v^{-1} \cdot u \tag{9}$$
- *calculations* in Lie fields: if $[a, b] \equiv ab - ba = c$ holds for some $a, b, c \in D(\mathfrak{g})$, then
$$[a, b^{-1}] = -b^{-1}cb^{-1} \tag{10}$$

Lie Field Extensions

- let $D(\mathfrak{g})[\theta]$ denote the *algebra of all polynomials* in the variable θ, satisfying $a\theta = \theta a$ for every $a \in D(\mathfrak{g})$, *with coefficients from* $D(\mathfrak{g})$
- choose a polynomial
$$\varphi(\theta) = \sum_{i=1}^{n} a_i \theta^i \in D(\mathfrak{g})[\theta]$$
such that $a_i \in Z(D(\mathfrak{g}))$, $1 \leq i \leq n$, and $a_n = 1$,
that is irreducible in $D(\mathfrak{g})$
- the **extended Lie field** is then defined to be the algebra
$$D_\varphi(\mathfrak{g}) \equiv \{\psi \in D(\mathfrak{g})[\theta];\ \deg \psi < n\}$$
with multiplication given by $\left\{\psi \cdot \tilde{\psi}\right\}(\theta) \equiv \psi(\theta)\tilde{\psi}(\theta) \mod \varphi(\theta)$

The Gelfand-Kirillov Conjecture

Definition

Let $\mathbb{F}$ be a field and let $m, r \in \mathbb{N}_0$. The **(extended) Weyl algebra** $\mathcal{W}_{m,r}(\mathbb{F})$ is defined to be the unital associative algebra over $\mathbb{F}$, generated by $q_1, \ldots, q_n, p_1, \ldots, p_n$ and $\theta_1, \ldots, \theta_r$ subject to the following relations:

$$p_i q_j - q_j p_i = \delta_{ij}, \quad 1 \leq i, j \leq m. \tag{11}$$

- let $D_{m,r}(\mathbb{F})$ denote the quotient field generated by $\mathcal{W}_{m,r}(\mathbb{F})$

Conjecture (Hypothèse fondamentale)

If $\mathfrak{g}$ is an algebraic Lie algebra over a commutative field $\mathbb{F}$ of characteristic zero, then there are $m, r \in \mathbb{N}_0$ such that $D(\mathfrak{g}) \cong D_{m,r}(\mathbb{F})$.

- m and r are uniquely determined
- $m = \frac{1}{2}(\dim_{\mathbb{F}} \mathfrak{g} - r)$

The Simply Connected Poincarè Group and Its Lie Algebra

Definition

Let $n \in \mathbb{N}$ and denote $g = \mathrm{diag}(1, -1, \dots, -1) \in \mathbb{R}^{n,n}$. We define the **simply connected Poincarè group** to be $\mathcal{P}_n = SO_0(1, n-1) \ltimes T^n$, where

$$SO_0(1, n-1) = \left\{ \Lambda \in \mathbb{R}^{n,n} \middle| \Lambda^T g \Lambda = g,\, \det \Lambda = 1,\, \Lambda_{00} \geq 1 \right\}, \qquad (12)$$

and

$$T^n = \{a | a \in \mathbb{R}^n\} \qquad (13)$$

- the Lie algebra $\mathfrak{p}_n$ of $\mathcal{P}_n$ (the **Poincarè algebra**) is generated by P_μ, $0 \leq \mu \leq n-1$ and $L_{\mu\nu}$, $0 \leq \mu < \nu \leq n-1$ subject to the following:

$$\begin{aligned} [L_{\mu\nu}, L_{\rho\sigma}] &= -(g_{\mu\rho} L_{\nu\sigma} - g_{\mu\sigma} L_{\nu\rho} + g_{\nu\sigma} L_{\mu\rho} - g_{\nu\rho} L_{\mu\sigma}), \\ [L_{\mu\nu}, P_\rho] &= -(g_{\mu\rho} P_\nu - g_{\nu\rho} P_\mu), \end{aligned} \qquad (14)$$

where $L_{\mu\nu} = -L_{\nu\mu}$

The Case $\mathcal{P}_2$

- note that $\mathfrak{p}_n$ is considered to be a *real* Lie algebra for all $n \in \mathbb{N}$
- for $n = 2$, $\mathfrak{p}_2$ is generated by P_0, P_1 and L_{01} due to

$$[L_{01}, P_0] = -P_1, \quad [L_{01}, P_1] = -P_0, \quad [P_0, P_1] = 0 \tag{15}$$

- in this case we have

$$r = 3 - \operatorname{rank}_{D(\mathfrak{p}_2)} \begin{pmatrix} 0 & -P_1 & -P_0 \\ P_1 & 0 & 0 \\ P_0 & 0 & 0 \end{pmatrix} = 1 \quad \text{and} \quad m = \frac{1}{2}(3-1) = 1$$

- thus we are searching for an isomorphism

$$\phi\colon \mathfrak{p}_2 \to D_{1,1}(\mathbb{R}),$$

  where $D_{1,1}(\mathbb{R})$ is generated by p, q and θ with $[p, q] = 1$ and zero otherwise

The Case $\mathcal{P}_2$

- since $P_0 - P_1 = [L_{01}, P_0 - P_1] = L_{01}(P_0 - P_1) - (P_0 - P_1)L_{01}$,

$$\begin{aligned} 1 &= L_{01} - (P_0 - P_1)L_{01}(P_0 - P_1)^{-1} \\ &= L_{01} - L_{01} - [P_0 - P_1, L_{01}(P_0 - P_1)^{-1}] \\ &= [L_{01}(P_0 - P_1)^{-1}, P_0 - P_1], \end{aligned}$$

  and similarly $1 = [(P_0 - P_1)^{-1}L_{01}, P_0 - P_1]$

- together we have

$$1 = \left[\frac{1}{2}\left\{L_{01}(P_0 - P_1)^{-1} + (P_0 - P_1)^{-1}L_{01}\right\}, P_0 - P_1\right] \tag{16}$$

- now we make use of $[L_{01}, (P_0 - P_1)^{-1}] = -(P_0 - P_1)^{-1}$ to write

$$\frac{1}{2}\left\{L_{01}(P_0 - P_1)^{-1} + (P_0 - P_1)^{-1}L_{01}\right\} = (P_0 - P_1)^{-1}\left(L_{01} - \frac{1}{2}\right)$$

The Case $\mathcal{P}_2$

- let us denote

$$\hat{p} := (P_0 - P_1)^{-1}\left(L_{01} - \frac{1}{2}\right), \tag{17}$$

$$\hat{q} := P_0 - P_1 \tag{18}$$

- the centre of $U(\mathfrak{p}_n)$ is generated by the only Casimir element

$$M^2 := P_0^2 - P_1^2 = (P_0 - P_1)(P_0 + P_1) \tag{19}$$

- using M^2, the relations (17) and (18) can be converted as

$$L_{01} = \hat{q}\hat{p} + \frac{1}{2}, \tag{20}$$

$$P_0 = \frac{1}{2}\left(\hat{q} + \hat{q}^{-1}M^2\right), \tag{21}$$

$$P_1 = \frac{1}{2}\left(-\hat{q} + \hat{q}^{-1}M^2\right) \tag{22}$$

The Case $\mathcal{P}_2$

- we define ϕ as follows:

$$\phi(L_{01}) := qp + \frac{1}{2}, \tag{23}$$

$$\phi(P_0) := \frac{1}{2}\left(q + q^{-1}\theta\right), \tag{24}$$

$$\phi(P_1) := \frac{1}{2}\left(-q + q^{-1}\theta\right) \tag{25}$$

- for the inverse $\phi^{-1}\colon D_{1,1}(\mathbb{R}) \to \mathfrak{p}_2$ we then have

$$\phi^{-1}(p) := (P_0 - P_1)^{-1}\left(L_{01} - \frac{1}{2}\right), \tag{26}$$

$$\phi^{-1}(q) := P_0 - P_1, \tag{27}$$

$$\phi^{-1}(\theta) := M^2 \tag{28}$$

The Case $\mathcal{P}_2$

- given a representation of $\tilde{U}$ of $D_{1,1}(\mathbb{R})$ on a Hilbert space $\mathcal{H}$, we may now construct the following representation U of $\mathfrak{p}_2$ (on $\mathcal{H}$):

$$U(X)\psi := \tilde{U}(\phi(X))\psi, \quad X \in \mathfrak{p}_2, \, \psi \in \mathcal{H} \tag{29}$$

- let us choose the family of representations $\tilde{U}_{m^2}$, $m^2 \in \mathbb{R}$, acting on $L^2(\mathbb{R}, x)$ defined as follows:

$$\tilde{U}(q) := ix, \quad , \tilde{U}(p) := -i\partial_x, \quad \tilde{U}(\theta) := m^2 I \tag{30}$$

- then by (29) we have

$$U_{m^2}(L_{01}) := x\partial_x + \frac{1}{2}, \tag{31}$$

$$U_{m^2}(P_0) := \frac{i}{2}\left(x - \frac{m^2}{x}\right), \tag{32}$$

$$U_{m^2}(P_1) := -\frac{i}{2}\left(x + \frac{m^2}{x}\right) \tag{33}$$

The Case $\mathcal{P}_2$

- the representation of $\mathfrak{p}_2$ can be, at least locally, integrated into the following representation of the Lie group $\mathcal{P}_2$ (on $L^2(\mathbb{R}, x)$):

$$\begin{aligned}\{U_{m^2}(t_1,t_2,t_3)\psi\}(x) &\equiv \left\{U^{(2)}_{m^2}(t_2)U^{(3)}_{m^2}(t_3)U^{(1)}_{m^2}(t_1)\psi\right\}(x)\\ &= \exp\left\{\frac{t_1}{2}+\frac{it_2}{2}\left(x-\frac{m^2}{x}\right)-\frac{it_3}{2}\left(x+\frac{m^2}{x}\right)\right\}\psi\left(e^{t_1}x\right),\end{aligned} \tag{34}$$

where

$$\left\{U^{(1)}_{m^2}(t)\psi\right\}(x) \equiv \{\exp(tL_{01})\psi\}(x) = e^{\frac{t}{2}}\psi(e^t x), \tag{35}$$

$$\left\{U^{(2)}_{m^2}(t)\psi\right\}(x) \equiv \{\exp(tP_0)\psi\}(x) = e^{\frac{it}{2}\left(x-\frac{m^2}{x}\right)}\psi(x), \tag{36}$$

$$\left\{U^{(3)}_{m^2}(t)\psi\right\}(x) \equiv \{\exp(tP_0)\psi\}(x) = e^{-\frac{it}{2}\left(x+\frac{m^2}{x}\right)}\psi(x), \tag{37}$$

for any $\psi \in L^2(\mathbb{R}, x)$

The Case $\mathcal{P}_2$

- the representation (34) is reducible since it has two invariant subspace, namely $L^2(\mathbb{R}^+)$ and $L^2(\mathbb{R}^-)$
- thus let us denote
$$\left\{ U^{\pm}_{m^2}(t_1, t_2, t_3)\psi \right\}(x) \equiv e^{\frac{t_1}{2} + \frac{it_2}{2}\left(x - \frac{m^2}{x}\right) - \frac{it_3}{2}\left(x + \frac{m^2}{x}\right)} \psi\left(e^{t_1}x\right), \quad (38)$$
for any $\psi \in L^2(\mathbb{R}^{\pm}, x)$
- comparing this result with the "standard" representations constructed due to the so-called *Mackey theory*, one can claim the following:

(38) **defines mutually non-equivalent irreducible representations of the Poincarè group** $\mathcal{P}_2$

That's all Folks!