Pseudospectrum and spectral stability of the discrete Schrödinger operator with a complex step potential

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Overview

1. **Preliminaries**
2. **Pseudospectrum**
3. **Asymptotic behavior**
4. **Spectral stability**

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Setting

- Consider the Hilbert space

$$\ell^2(\mathbb{Z}) = \left\{ \{x_n\}_{n\in\mathbb{Z}} \;\middle|\; \sum_{n\in\mathbb{Z}} |x_n|^2 < +\infty \right\}.$$

- We study the operator H_α on $\ell^2(\mathbb{Z})$ given by

$$(H_\alpha x) = \begin{cases} x_{n-1} + x_{n+1} & n < 0, \\ x_{n-1} + \alpha x_n + x_{n+1} & n \geq 0, \end{cases}$$

  where α is a free complex parameter.
- The spectrum of H_α

$$\sigma(H_\alpha) = \sigma_{\mathrm{c}}(H_\alpha) = \sigma_{\mathrm{ess}}(H_\alpha) = [-2, 2] \cup [-2 + \alpha, 2 + \alpha].$$

Joukowsky transform

$$\lambda = \xi + \xi^{-1}$$

- Bijective map between

$$\mathbb{C} \setminus [-2, 2] \qquad \longleftrightarrow \qquad \left\{\xi \in \mathbb{C} \;\middle|\; 0 < |\xi| < 1\right\}$$

- We also define $\lambda - \alpha = \eta + \eta^{-1}$

Green kernel

The resolvent operator $(H_\alpha - \lambda)^{-1} = G(\lambda)$ and reads

$$G_{m,n}(\lambda) = \begin{cases} \frac{\eta^{m+n} - \eta^{|m-n|}}{\eta^{-1} - \eta} + \frac{\eta^{m+n}}{w} & m, n \geq 0, \\ \frac{1}{w}\eta^m \xi^{-n} & m \geq 0, n < 0, \\ \frac{\xi^{|m-n|} - \xi^{-m-n}}{\xi - \xi^{-1}} + \frac{\xi^{-m-n}}{w} & m, n < 0, \\ \frac{1}{w}\eta^n \xi^{-m} & m < 0, n \geq 0, \end{cases}$$

where w is the Wronskian and reads $w = \xi - \eta^{-1}$.

Pseudospectrum

Definition

Let $A \in \mathscr{B}(\mathscr{H})$ and $\varepsilon > 0$ be arbitrary. The ε-*pseudospectrum* of operator A is defined as the set

$$\sigma_\varepsilon(A) := \sigma(A) \cup \left\{\lambda \in \mathbb{C} \setminus \sigma(A) \;\middle|\; \|(A-\lambda)^{-1}\| \geq \varepsilon^{-1}\right\}.$$

- Each $\sigma_\varepsilon(A)$ is a nonempty subset of $\mathbb{C}$.
- Any bounded connected component of $\sigma_\varepsilon(A)$ has a nonempty intersection with $\sigma(A)$.
- The pseudospectra are strictly nested supersets of the spectrum.

If $\lambda \in \sigma(A)$, we write $\left\|(A-\lambda)^{-1}\right\| = \infty$.

Self-adjoint case

Proposition

Let $A \in \mathscr{B}(\mathscr{H})$ be a self-adjoint operator and $\lambda \in \varrho(A)$, then

$$\|(A-\lambda)^{-1}\| = \frac{1}{\operatorname{dist}(\sigma(A),\lambda)}.$$

Let α be a real number. Then

$$\varepsilon > 0: \qquad \sigma_\varepsilon(H_\alpha) = [-2,2] \cup [-2+\alpha, 2+\alpha] \ + \ D(\varepsilon),$$

where $D(\varepsilon) := \{z \in \mathbb{C} \mid |z| < \varepsilon\}$.

Estimates of resolvent operator's norm

Proposition

Let $A \in \mathscr{B}(\mathscr{H})$ and $\lambda \in \rho(A)$. Then

$$\frac{1}{\operatorname{dist}\big(\lambda, \sigma(A)\big)} \leq \left\|(A-\lambda)^{-1}\right\|.$$

Proposition (T. Kato, 1995)

Let $A \in \mathscr{B}(\mathscr{H})$ and $\lambda \in \mathbb{C} \setminus \overline{\mathrm{Num}(A)}$. Then

$$\forall \lambda \in \rho(A) \;:\; \left\|(A-\lambda)^{-1}\right\| \leq \frac{1}{\operatorname{dist}\big(\lambda, \mathrm{Num}(A)\big)}.$$

- The numerical range of operator H_α reads

$$\mathrm{Num}(H_\alpha) = [-2,2] + [0,\alpha].$$

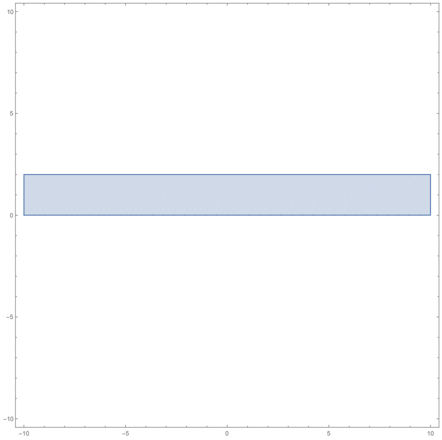
- On the set

$$\Omega(\alpha) := \big\{ z \in \mathbb{C} \;\big|\; \mathrm{dist}\big(\lambda, \mathrm{Num}(H_\alpha)\big) = \mathrm{dist}\big(\lambda, \sigma(H_\alpha)\big) \big\}$$

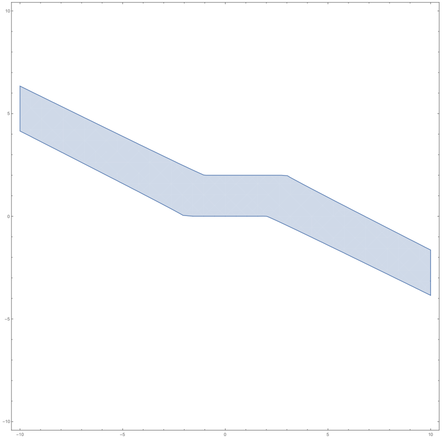
we can describe the ε-spectrum explicitly

$$\Omega(\alpha) \cap \sigma_\varepsilon(H_\alpha) \;=\; \Omega(\alpha) \cap \big(\sigma(H_\alpha) + D(\varepsilon)\big).$$

- On $\mathbb{C} \setminus \Omega$ we estimate the resolvent operator's norm; thus, defining a subset and a superset of the ε-speudospectrum.



(a) $\alpha = 2\mathrm{i}$



(b) $\alpha = 2\mathrm{i} + 1$

Figure: Visualization of the set $\Omega(\alpha)$.

Lower bound

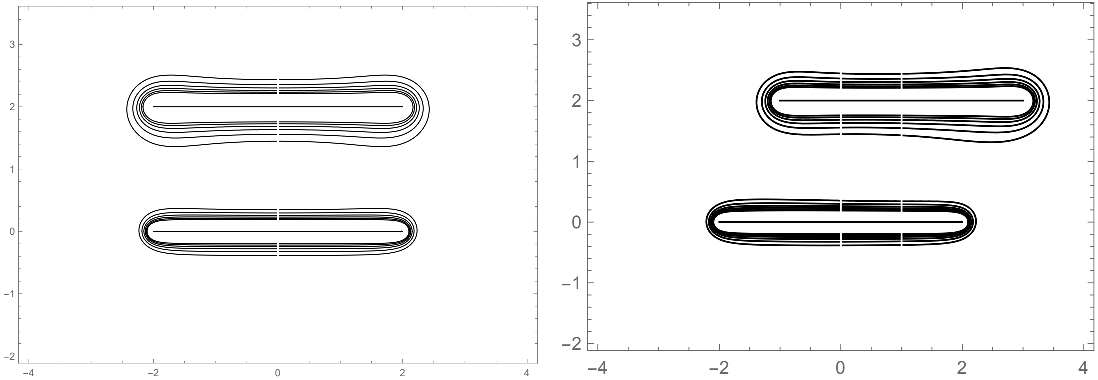
- Define a test vector

$$\psi_n := \begin{cases} \xi^{-n} & n \leq 0, \\ \eta^n & n \geq 0. \end{cases}, \qquad \|\psi\|^2 = \frac{1 - |\xi\eta|^2}{(1 - |\xi|^2)(1 - |\eta|^2)}$$

- From the definition of operator norm we have

$$\|(H_\alpha - \lambda)^{-1}\| \geq \frac{\|(H_\alpha - \lambda)^{-1}\psi\|}{\|\psi\|} =: L(\lambda).$$

$$\|(H_\alpha - \lambda)^{-1}\| \geq \frac{\sqrt{(1-|\xi|^2)(1-|\eta|^2)}}{\sqrt{1-|\xi\eta|^2}} \frac{|1-\xi\eta|}{|w||1-\xi^2||1-\eta^2|} \sqrt{\frac{|1+\xi\eta|^2(1-|\xi\eta|^2)}{(1-|\xi|^2)(1-|\eta|^2)} + 2\big(\mathrm{Re}(1+\xi\eta)\mathrm{Re}(\xi(\eta^{-1}-\eta)) + \mathrm{Im}(1+\xi\eta)\mathrm{Im}(\xi(\eta^{-1}-\eta))\big)\frac{|\xi|^2}{(1-|\xi|^2)^2} \overline{+2\big(\mathrm{Re}(1+\xi\eta)\mathrm{Re}(\xi(\xi^{-1}-\xi) + \mathrm{Im}(1+\xi\eta)\mathrm{Im}(\xi(\xi^{-1}-\xi)\big)\frac{|\eta|^2}{(1-|\eta|^2)^2} + |\xi(\eta^{-1}-\eta)|^2\frac{|\xi|^2(1+|\xi|^2)}{(1-|\xi|^2)^3} + |\xi^{-1}-\xi|^2\frac{|\eta|^2(1+|\eta|^2)}{(1-|\eta|^2)^3}}}.$$



(a) $\alpha = 2\mathrm{i}$ (b) $\alpha = 2\mathrm{i} + 1$

Figure: Visualization of $\left\{\lambda \in \mathbb{C} \;\middle|\; \|L(\lambda)\| \geq \varepsilon^{-1}\right\}$ for various ε.

Schur test

Theorem

Let $A = (a_{i,j})_{i,j\in\mathbb{Z}}$ be a doubly infinite matrix of complex entries. If there exists a positive number p_j for all $j \in \mathbb{Z}$ such that

$$\sup_{j\in\mathbb{Z}} \frac{1}{p_j} \sum_{i\in\mathbb{Z}} |a_{i,j}| p_i =: \alpha < \infty \qquad \& \qquad \sup_{i\in\mathbb{Z}} \frac{1}{p_i} \sum_{j\in\mathbb{Z}} |a_{i,j}| p_j =: \beta < \infty,$$

then $A \in \mathscr{B}(\ell^2(\mathbb{Z}))$ and $\|A\| \leq \sqrt{\alpha\beta}$.

Upper bound

- The Schur test implies

$$\|(H_\alpha - \lambda)^{-1}\| \leq \sup_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \left|G_{m,n}(\lambda)\right|.$$

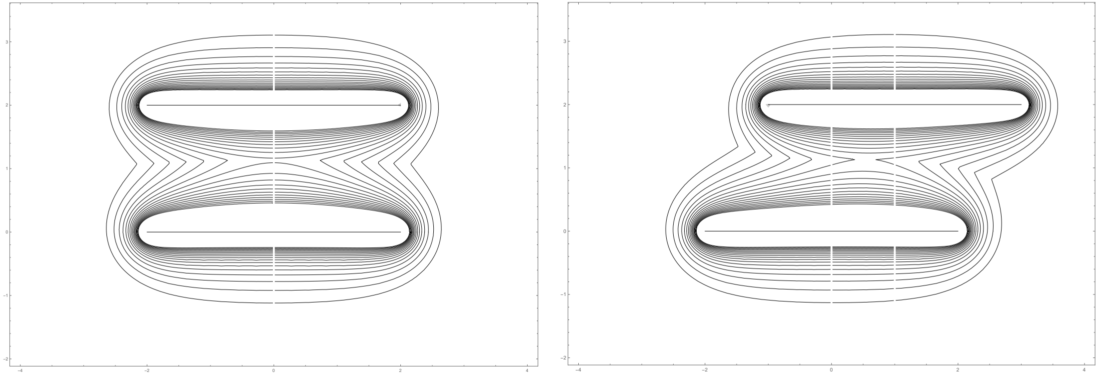
- If we define

$$U_1(\lambda) := \frac{|\xi\eta|}{|w|(1-|\eta|)} + \frac{1}{|w|(1-|\xi|)}\left(\left|\frac{\eta^{-1}-\xi^{-1}}{\xi-\xi^{-1}}\right||\xi| + (1+|\xi|)\left|\frac{\eta^{-1}-\xi}{\xi^{-1}-\xi}\right|\right),$$

$$U_2(\lambda) := \frac{|\xi|}{|w|(1-|\xi|)} + \frac{1}{|w|(1-|\eta|)}\left(\left|\frac{\xi-\eta}{\eta^{-1}-\eta}\right| + (1+|\eta|)\left|\frac{\xi-\eta^{-1}}{\eta-\eta^{-1}}\right|\right),$$

the upper bound of the resolvent operator's norm reads

$$\|(H_\alpha - \lambda)^{-1}\| \leq \max\left\{U_1(\lambda), U_2(\lambda)\right\} =: U(\lambda).$$



(a) $\alpha = 2\mathrm{i}$ (b) $\alpha = 2\mathrm{i} + 1$

Figure: Visualization of $\left\{\lambda \in \mathbb{C} \;\middle|\; U(\lambda) \;\geq \varepsilon^{-1}\right\}$ for various ε.

Global estimates of ε-pseudospectrum

We derived

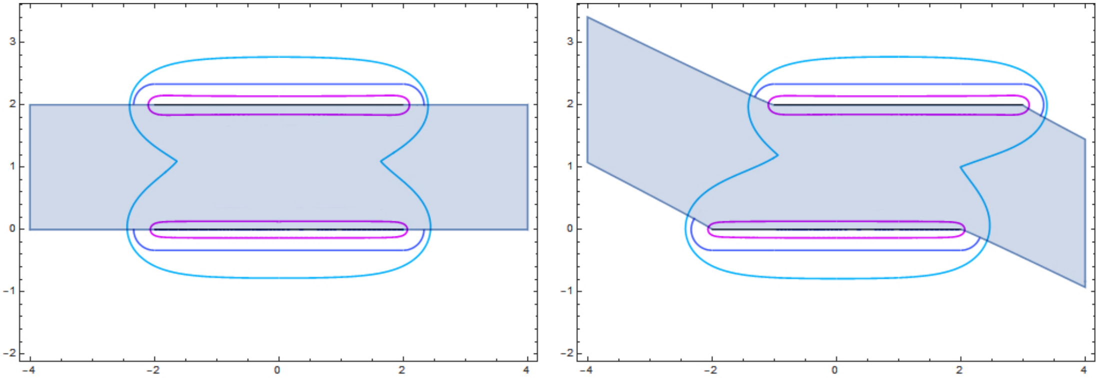
$$L(\lambda) \leq \|(H_\alpha - \lambda)^{-1}\| \leq U(\lambda)$$

Proposition

The ε-pseudospectrum of H_α satisfies these inclusions

$$\left\{\lambda \in \mathbb{C} \;\middle|\; L(\lambda) \;\geq \varepsilon^{-1}\right\} \subset \sigma_\varepsilon(H_\alpha) \subset \left\{\lambda \in \mathbb{C} \;\middle|\; U(\lambda) \;\geq \varepsilon^{-1}\right\}.$$

In the following images, we only show the boundaries of these sets.



(a) $\alpha = 2\mathrm{i}$ (b) $\alpha = 2\mathrm{i} + 1$

Figure: Visualization of $\frac{1}{3}$-pseudospectrum.

Asymptotics

- The spectrum of H_α has to connected components $\mathrm{Im}\alpha \neq 0$

$$\begin{aligned} \lambda \longrightarrow [-2,2] \quad &\longleftrightarrow \quad |\xi| \longrightarrow 1 \quad : \quad \varepsilon \to 0_+, \text{ where } \xi = (1-\varepsilon)e^{i\phi} \\ \lambda \longrightarrow [-2+\alpha, 2+\alpha] \quad &\longleftrightarrow \quad |\eta| \longrightarrow 1 \quad : \quad \varepsilon \to 0_+, \text{ where } \eta = (1-\varepsilon)e^{i\phi} \end{aligned}$$

- We decompose the resolvent operator

$$G(\lambda) = \begin{pmatrix} G^{--} & G^{-+} \\ G^{+-} & G^{++} \end{pmatrix} = \begin{pmatrix} G^{--} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & G^{-+} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ G^{+-} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & G^{++} \end{pmatrix}$$

- We estimate the norm with triangle inequality from above and below
- We show that all but one term is asymptotically insignificant

Proposition

The asymptotic behavior of the resolvent operator's norm reads

$$\frac{1}{\varepsilon}\frac{\sqrt{2}}{4|\sin\phi|} + \mathcal{O}(1) \leq \|(H_\alpha - \lambda)^{-1}\| \leq \frac{1}{\varepsilon}\left(\frac{1}{\widehat{w}} + \frac{3}{2|\sin\phi|}\right) + \mathcal{O}(1) \qquad \text{as } |\xi|, |\eta| \to 1,$$

if it does not approach the spectrum along the line on which a given connected component resides, i.e. $\phi \neq 0, \pm\pi$. Otherwise, the norm follows

$$\frac{\sqrt{2}}{4\varepsilon^2} + \mathcal{O}(\varepsilon^{-1}) \leq \|(H_\alpha - \lambda)^{-1}\| \leq \frac{3}{2\varepsilon^2} + \mathcal{O}(\varepsilon^{-1}) \qquad \text{as } |\xi|, |\eta| \to 1.$$

$\widehat{w}$ denotes a constant which is a uniform lower bound of $|w| = |\xi - \eta^{-1}|$.

Birman–Schwinger principle

- Let $v = \{v_n\}_{n\in\mathbb{Z}}$. We denote

$$V_{1/2} := \operatorname{diag}\Big(\big\{\sqrt{|v_n|}\,\operatorname{sgn} v_n\big\}_{n\in\mathbb{Z}}\Big), \qquad |V|^{1/2} := \operatorname{diag}\Big(\big\{\sqrt{|v_n|}\big\}_{n\in\mathbb{Z}}\Big).$$

Theorem

Let $H \in \mathscr{B}(\ell^2(\mathbb{Z}))$, $\lambda \in \rho(H)$, $v \in \ell^1(\mathbb{Z})$, then for $K(\lambda) := V_{1/2}(H-\lambda)^{-1}|V|^{1/2}$ we have

$$\lambda \in \sigma_{\mathrm{p}}(H+V) \qquad \Longleftrightarrow \qquad -1 \in \sigma_{\mathrm{p}}(K(\lambda)).$$

Weak coupling of H_0

Consider V to be strictly real-valued.

Proposition

Let $v \in \ell^1(\mathbb{Z})$ such that $\sum_{n\in\mathbb{Z}} n^2|v_n| \leq +\infty$. Then for all $\varepsilon > 0$

$$\sum v_n < 0 \quad \implies \quad H_0 + \varepsilon V \text{ has an eigenvalue } < -2,$$

$$\sum v_n > 0 \quad \implies \quad H_0 + \varepsilon V \text{ has an eigenvalue } > 2.$$

Moreover, if $\varepsilon < \left(\sum_{n\in\mathbb{Z}} n^2|v_n| + |v_0|\right)^{-1}$, the eigenvalue is unique.

This also follows from a more general result from [1].

[1] S. Yu. Kholmatova, S. N.Lakaevb, F. M.Almuratovb, 2021

$$(H_0 - \lambda)^{-1}_{m,n} = \frac{k^{|m-n|}}{k - k^{-1}}, \qquad m, n \in \mathbb{Z}, \quad 0 < |k| < 1$$

$$K(\lambda) = L(\lambda) + M(\lambda)$$

$$L(\lambda)_{m,n} := \sqrt{|v_m|}\, \frac{1}{k - k^{-1}}\, \sqrt{|v_n|}\, \mathrm{sgn}\, v_n$$

$$M(\lambda)_{m,n} := \sqrt{|v_m|}\, \frac{k^{|m-n|} - 1}{k - k^{-1}}\, \sqrt{|v_n|}\, \mathrm{sgn}\, v_n$$

$$(I - \varepsilon K(\lambda))^{-1} = \Big(I + (I + \varepsilon M(\lambda))^{-1} \varepsilon L(\lambda)\Big)^{-1} (I + \varepsilon M(\lambda))^{-1}$$

$k + k^{-1} = \lambda \in \sigma_{\mathrm{p}}(H_\varepsilon) \iff$ there is a solution k to the equation below

$$k - k^{-1} = -\varepsilon \langle v_{1/2}, (I + \varepsilon M(\lambda))^{-1} |v|^{1/2} \rangle$$

Weak coupling of $-\mathrm{d}^2/\mathrm{d}x^2$

We compare the result in the discrete setting with the one in the continuous setting.

Theorem (B. Simon, 1976)

Let $V \in L^1(\mathbb{R}, (1 + x^2)\mathrm{d}x)$. Then $-\mathrm{d}^2/\mathrm{d}x^2 + \varepsilon V$ has a negative eigenvalue for all $\varepsilon > 0$ if and only if

$$\int V(x)\mathrm{d}x \leq 0.$$

If $\varepsilon > 0$ is sufficiently small, the eigenvalue is unique.

Spectral stability of H_α

- We showed that there exists a doubly infinite matrix M such that

$$\forall\lambda \in \mathbb{C},\ \forall m, n \in \mathbb{Z} \ : \quad |G_{m,n}(\lambda)| \leq M_{m,n}.$$

- We showed that if $v \in \ell^1(\mathbb{Z})$ satisfies $\sum m^2 |v_m| < +\infty$, then

$$\|K(\lambda)\| \leq \frac{2}{\widehat{w}}\Big(\sum_{m\in\mathbb{Z}} m^2 |v_m| + |v_0|\Big).$$

- The operator $K(\lambda)$ scales proportionally to the potential V.
- If the norm of V is so small that $\|K(\lambda)\| < 1$, the Birman–Schwinger principle implies that λ is not an eigenvalue of $H_\alpha + V$.

Spectral stability of H_α

Theorem

Let $v \in \ell^1(\mathbb{Z})$ satisfy

$$\sum_{m\in\mathbb{Z}} m^2 |v_m| < \infty.$$

If we take $\varepsilon > 0$ so small that

$$\varepsilon < \frac{\widehat{w}}{2}\Big(\sum_{m\in\mathbb{Z}} m^2 |v_m| + |v_0|\Big)^{-1},$$

then the potential εV does not change the spectrum, i.e.

$$\sigma(H_\alpha) = \sigma(H_\alpha + \varepsilon V).$$

Comparison to the continuous case

Consider the operator H on $L^2(\mathbb{R})$ defined by

$$H := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mathrm{i}\,\mathrm{sgn}\,(x), \qquad \mathrm{Dom}(H) := W^{2,2}(\mathbb{R}).$$

Theorem (R. Henry, D. Krejčiřík, 2017)

Let $V \in L^1(\mathbb{R}, (1+x^2)\mathrm{d}x)$ and $\mathcal{S} = [0, +\infty) + (-\mathrm{i}, \mathrm{i})$. There exists a positive constant C (independent of V and ε) such that, whenever

$$\varepsilon \int_{\mathbb{R}} V(x)(1+x^2)\mathrm{d}x \leq \frac{1}{C},$$

we have

$$\sigma_{\mathrm{p}}(H + \varepsilon V) \subset \overline{\mathcal{S}} \cap \left\{ z \in \mathbb{C} \;\middle|\; \mathrm{Re}\, z \geq \frac{C}{\varepsilon^2 \|V\|^2_{L^1(\mathbb{R})}} \right\}.$$

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