Pseudospectra in an operator-theoretic description of black holes

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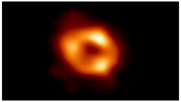

Figure: Sagittarius A* by EHT [5]

- Wave equation

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^{*2}} + V_l\right)\phi_{lm} = 0$$

Black hole quasinormal modes [1]

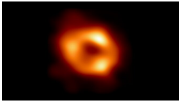

Figure: Sagittarius A* by EHT [5]

- Wave equation

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^{*2}} + V_l\right)\phi_{lm} = 0$$

- Change of coordinates

$$\begin{cases} r^* &= 2M\left(\frac{1}{x} + \ln(1-x) - \ln x\right), \\ t &= 4M\tau - 2M\left(\ln x + \ln(1-x) - \frac{1}{x}\right), \end{cases}$$

$$r^* \in (-\infty, \infty) \longleftrightarrow x \in (0,1)$$

- Wave equation

$$-\partial_\tau^2\phi_{lm} + L_1\phi_{lm} + L_2\partial_\tau\phi_{lm} = 0,$$

$$L_1 = \frac{1}{1+x}\left(\partial_x(x^2(1-x)\partial_x) - \frac{(4M)^2 V_l}{4x^2(1-x)}\right),$$

$$L_2 = \frac{1}{1+x}\left((1-2x^2)\partial_x - 2x\right)$$

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- Matrix operator

$$\partial_\tau u_{lm} = \mathrm{i} L u_{lm},$$
$$L = \frac{1}{\mathrm{i}}\begin{pmatrix} 0 & I \\ L_1 & L_2 \end{pmatrix}, \quad u_{lm} = \begin{pmatrix} \phi_{lm} \\ \psi_{lm} \end{pmatrix}, \quad \psi_{lm} = \partial_\tau \phi_{lm},$$
$$u_{lm}(\tau, x) \sim u_{lm}(x) e^{i\omega\tau}, \quad L u_{n,lm} = \omega_{n,lm} u_{n,lm}$$

Operator L

- Simplified version

$$L = \frac{1}{\mathrm{i}} \begin{pmatrix} 0 & I \\ L_1 & 0 \end{pmatrix},$$

$$L_1 = \frac{1}{w(x)} \left(\partial_x (p(x) \partial_x) - V(x) \right),$$

$$w(x) = 1 + x, \quad p(x) = x^2(1-x), \quad V(x) \in C^\infty,$$

$$V(x) \geq V_0 > 0, \quad x \in (0,1)$$

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- Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$

$$\|(u,v)\|^2_{\mathcal{H}} := \|u\|^2_{\mathcal{H}_1} + \|v\|^2_{\mathcal{H}_2}, \ u \in \mathcal{H}_1, \ v \in \mathcal{H}_2,$$

$$\mathcal{H}_1 = \mathcal{W} = \overline{C^\infty((0,1))}^{\|\cdot\|_{\mathcal{W}}}, \quad \mathcal{H}_2 = L^2_w = L^2((0,1), w(x)dx)$$

- Hilbert space $\mathcal{H}$

$$\begin{aligned}\langle (u,v), L(u,v)\ \rangle_{\mathcal{H}} &= \langle (u,v), -\mathrm{i}(v, L_1 u)\rangle_{\mathcal{H}} \\ &= -\mathrm{i}\left(\langle u,v\rangle_{\mathcal{W}} + \langle v, L_1 u\rangle_{L^2_w}\right) \\ &= -\mathrm{i}\left(\langle u,v\rangle_{\mathcal{W}} - \langle v', pu'\rangle_{L^2} - \langle v, Vu\rangle_{L^2}\right),\end{aligned}$$

$$\|u\|^2_{\mathcal{W}} := \left\|p^{1/2}u'\right\|^2_{L^2} + \left\|V^{1/2}u\right\|^2_{L^2},$$

$$\Rightarrow\ \langle (u,v), L(u,v)\ \rangle_{\mathcal{H}} = 2\Im(\langle u', pv'\rangle_{L^2} + \langle u, Vv\rangle_{L^2}) \in \mathbb{R},$$

$$\mathcal{H} = \mathcal{W} \oplus L^2_w, \quad \|(u,v)\|^2_{\mathcal{H}} = \|u\|^2_{\mathcal{W}} + \|v\|^2_{L^2_w}$$

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- $\mathcal{W}$ is boundedly embedded in L^2_w

$$\|u\|^2_{\mathcal{W}} = \|p^{1/2}u'\|^2_{L^2} + \|V^{1/2}u\|^2_{L^2} \geq V_0\|u\|^2_{L^2} \approx \|u\|^2_{L^2_w}$$

Domain of L

- $\mathrm{Dom}(L) = \mathrm{Dom}(L_1) \times \mathrm{Dom}(I)$

$$
\begin{aligned}
I : L^2_w &\longrightarrow \mathcal{W}, \quad \mathrm{Dom}(I) = \mathcal{W}, \\
L_1 : \mathcal{W} &\longrightarrow L^2_w, \quad \mathrm{Dom}(L_1) = ?
\end{aligned}
$$

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- Lax-Milgram theorem

$$T : L^2_w \longrightarrow L^2_w, \quad T = -L_1, \quad \mathrm{Dom}(T) = \mathrm{Dom}(L_1),$$
$$t(u,u) = \langle u, Tu \rangle_{L^2_w} = \|u\|^2_{\mathcal{W}}, \quad \|u\|^2_t := t(u,u) + \|u\|^2_{L^2_w},$$
$$\mathrm{Dom}(t) = \mathcal{V} = \overline{C^\infty}^{\|\cdot\|_t} \subseteq L^2_w \subseteq \mathcal{V}^*,$$
$$\|\cdot\|_t \approx \|\cdot\|_{\mathcal{W}}, \ \mathcal{V} = \mathcal{W} \text{ as sets}$$

- Bounded: $|t(u,v)| \leq t(u,u) \cdot t(v,v) \leq \|u\|_t \|v\|_t$
- Coercive:

$$|t(u,u)| \geq \|p^{1/2}u'\|_{L^2}^2 + \frac{1}{2}\|V^{1/2}u\|_{L^2}^2 + \frac{1}{2}V_0\|u\|_{L^2}^2$$
$$\gtrsim \|p^{1/2}u'\|_{L^2}^2 + \|V^{1/2}u\|_{L^2}^2 + \|u\|_{L^2}^2 \gtrsim \|u\|_t^2$$

- Distributional operator: $\hat{T} \in \mathscr{B}(\mathcal{V}, \mathcal{V}^*)$, $\hat{T}u := t(\cdot, u)$, $T := \hat{T}|_{\mathrm{Dom}(T)}$, $\mathrm{Dom}(T) := \{u \in \mathcal{V} : \hat{T}u \in L^2_w\}$ dense in $\mathcal{V}$ and L^2_w

$$\boxed{\mathrm{Dom}(L) = \mathrm{Dom}(T) \times \mathcal{W}}$$

Spectrum of L

- Motivation:

$$\frac{1}{\mathrm{i}}\begin{pmatrix} 0 & I \\ L_1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix} \Rightarrow \begin{cases} v = \mathrm{i}\lambda u \\ L_1 u = \mathrm{i}\lambda v \end{cases}$$

$$\Longrightarrow L_1 u = -\lambda^2 u, \quad \text{"}\lambda \in \sigma(L) \overset{?}{\longleftrightarrow} -\lambda^2 \in \sigma(L_1)\text{"}$$

Theorem:

$\forall \lambda \neq 0: \ \lambda \in \rho(L) \Longleftrightarrow \lambda^2 \in \rho(T)$

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Theorem:

$\forall \lambda \neq 0: \; \lambda \in \rho(L) \Longleftrightarrow \lambda^2 \in \rho(T)$

- Schur complement: $S_2 = -\lambda - \frac{1}{\lambda} L_1 = \frac{1}{\lambda}(T - \lambda^2)$

$$(L-\lambda)^{-1} = \begin{pmatrix} -\frac{1}{\lambda} + \frac{1}{\lambda}(T-\lambda^2)^{-1} T & -\mathrm{i}(T-\lambda^2)^{-1} \\ \mathrm{i}(T-\lambda^2)^{-1} T & \lambda (T-\lambda^2)^{-1} \end{pmatrix}$$

Spectral equivalence

$\lambda \in \rho(L) \Rightarrow \lambda^2 \in \rho(T)$:

- $(L-\lambda)^{-1} \in \mathscr{B}(\mathcal{W} \oplus L^2_w)$

$$\begin{cases} \pi_1 : \mathcal{W} \oplus L^2_w \longrightarrow \mathcal{W} : (u,v) \longmapsto u, \ (\pi_2 : (u,v) \longmapsto v), \\ \pi_1^* : \mathcal{W} \longrightarrow \mathcal{W} \oplus L^2_w : u \longmapsto (u,0), \ (\pi_2^* : v \longmapsto (0,v)) \end{cases}$$

Spectral equivalence

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- $U_\lambda := \frac{1}{\lambda}\pi_2(L-\lambda)^{-1}\pi_2^* \in \mathscr{B}(L^2_w), \quad U_\lambda = (T-\lambda^2)^{-1}?$
- $v \in L^2_w$:

$$\begin{aligned} (0,v) = \pi_2^* v &= (L-\lambda)(L-\lambda)^{-1}\pi_2^* v = \\ &= \begin{pmatrix} -\lambda\pi_1(L-\lambda)^{-1}\pi_2^* v - \mathrm{i}\pi_2(L-\lambda)^{-1}\pi_2^* v \\ -\mathrm{i}L_1\pi_1(L-\lambda)^{-1}\pi_2^* v - \lambda\pi^2(L-\lambda)^{-1}\pi_2^* v) \end{pmatrix} \\ &\Rightarrow \begin{cases} U_\lambda v = \mathrm{i}\pi_1(L-\lambda)^{-1}\pi_2^* v \\ v = -L_1 U_\lambda v - \lambda^2 U_\lambda v = (T-\lambda^2)U_\lambda v \end{cases} \end{aligned}$$

- $v \in \mathrm{Dom}(T)$:

$$v = \frac{1}{\mathrm{i}}\pi_2(L-\lambda)^{-1}(L-\lambda)\left(\frac{1}{\lambda}v, \mathrm{i}v\right) = $$
$$= \frac{1}{\lambda}\pi_2(L-\lambda)^{-1}\pi_2^*(T-\lambda^2)v = U_\lambda(T-\lambda^2)v$$

$$\Rightarrow U_\lambda = (T-\lambda^2)^{-1} \Rightarrow \boxed{(T-\lambda^2)^{-1} \in \mathscr{B}(L_w^2)}$$

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- $\underline{\lambda^2 \in \rho(T) \Rightarrow \lambda \in \rho(L) :}$
  - $(T-\lambda^2)^{-1} \in \mathscr{B}(L^2_w)$

$$R_\lambda := \begin{pmatrix} -\frac{1}{\lambda} + \frac{1}{\lambda} I_{\mathcal{V} \longrightarrow \mathcal{W}} (\hat{T} - \lambda^2)^{-1} \hat{T} I_{\mathcal{W} \longrightarrow \mathcal{V}} & -\mathrm{i}(T-\lambda^2)^{-1} \\ \mathrm{i} I_{\mathcal{V} \longrightarrow L^2_w} (\hat{T} - \lambda^2)^{-1} \hat{T} I_{\mathcal{W} \longrightarrow \mathcal{V}} & \lambda (T-\lambda^2)^{-1} \end{pmatrix}$$

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  - Lemma 1: $(T-\lambda^2)^{-1} \in \mathscr{B}(L^2_w) \Longrightarrow (T-\lambda^2)^{-1} \in \mathscr{B}(L^2_w, \mathcal{W})$.
    Lemma 2: $(T-\lambda^2)^{-1} \in \mathscr{B}(L^2_w) \Longrightarrow (\hat{T}-\lambda^2)^{-1} \in \mathscr{B}(\mathcal{V}^*, \mathcal{V})$.

- Proof (L2):
  $\forall \lambda \in \mathbb{C} \setminus \{0\},\ \exists z_\lambda > 0:\ (\hat{T} - \lambda^2 + z_\lambda)^{-1} \in \mathscr{B}(\mathcal{V}^*, \mathcal{V}).$
  - L-M theorem applied to $t + (-\lambda^2 + z_\lambda)\|\cdot\|^2_{L^2_w}$,
    coercivity: $\forall u \in \mathcal{V}:\ |t(u,u) + (-\lambda^2 + z_\lambda)\|u\|^2_{L^2_w}| \geq$
    $\geq t(u,u) + (-\lambda^2 + z_\lambda)\|u\|^2_{L^2_w} = \|u\|^2_{\mathcal{W}} + (-\lambda^2 + z_\lambda)\|u\|^2_{L^2_w} \geq$
    $\geq \|u\|^2_t \quad$ if $\ z_\lambda \geq 1 + \lambda^2$

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    $\geq t(u,u) + (-\lambda^2 + z_\lambda)\|u\|^2_{L^2_w} = \|u\|^2_{\mathcal{W}} + (-\lambda^2 + z_\lambda)\|u\|^2_{L^2_w} \geq$
    $\geq \|u\|^2_t \quad \text{if} \ \ z_\lambda \geq 1 + \lambda^2$

- Resolvent identity:
  $(T - \lambda^2)^{-1} = (T - \lambda^2 + z_\lambda)^{-1} - z_\lambda(T - \lambda^2)^{-1}(T - \lambda^2 + z_\lambda)^{-1}$
  $\subseteq (\hat{T} - \lambda^2 + z_\lambda)^{-1} - z_\lambda(T - \lambda^2)^{-1} I_{\mathcal{V}\longrightarrow L^2_w}(\hat{T} - \lambda^2 + z_\lambda)^{-1}$
  $r_\lambda := (\hat{T} - \lambda^2 + z_\lambda)^{-1} - z_\lambda(T - \lambda^2)^{-1} I_{\mathcal{V}\longrightarrow L^2_w}(\hat{T} - \lambda^2 + z_\lambda)^{-1}$
  $r_\lambda \in \mathscr{B}(\mathcal{V}^*, \mathcal{V}), \quad r_\lambda = (\hat{T} - \lambda^2)^{-1}?$

- $u \in \mathrm{Dom}(T)$:
  $r_\lambda(\hat{T} - \lambda^2)u = r_\lambda(T - \lambda^2)u = (T - \lambda^2)^{-1}(T - \lambda^2)u = u$
- $u \in L^2_w$:

$$
\begin{aligned}
(\hat{T} - \lambda^2)r_\lambda u &= (\hat{T} - \lambda^2)(T - \lambda^2)^{-1}u = \\
&= (T - \lambda^2)(T - \lambda^2)^{-1}u = u
\end{aligned}
$$

$$\Rightarrow r_\lambda = (\hat{T} - \lambda^2)^{-1} \Rightarrow (\hat{T} - \lambda^2)^{-1} \in \mathscr{B}(\mathcal{V}^*, \mathcal{V})$$

- $u \in \mathrm{Dom}(T)$:
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$$

$$\Rightarrow r_\lambda = (\hat{T} - \lambda^2)^{-1} \Rightarrow (\hat{T} - \lambda^2)^{-1} \in \mathscr{B}(\mathcal{V}^*, \mathcal{V})$$

- $R_\lambda \in \mathscr{B}(\mathcal{W} \oplus L^2_w)$

$$R_\lambda = \begin{pmatrix} -\frac{1}{\lambda} + \frac{1}{\lambda} I_{\mathcal{V} \longrightarrow \mathcal{W}} (\hat{T} - \lambda^2)^{-1} \hat{T} I_{\mathcal{W} \longrightarrow \mathcal{V}} & -\mathrm{i}(T - \lambda^2)^{-1} \\ \mathrm{i} I_{\mathcal{V} \longrightarrow L^2_w} (\hat{T} - \lambda^2)^{-1} \hat{T} I_{\mathcal{W} \longrightarrow \mathcal{V}} & \lambda (T - \lambda^2)^{-1} \end{pmatrix}$$

- $(u, v) \in \mathcal{W} \oplus L^2_w,\ \ R_\lambda(u, v) \in \mathrm{Dom}(L)$:
  - $\pi_1 R_\lambda(u, v) = -\frac{1}{\lambda} u + \frac{1}{\lambda}(\hat{T} - \lambda^2)^{-1} \hat{T} u - \mathrm{i}(\hat{T} - \lambda^2)^{-1} v \ \in \mathcal{W}$
    $\hat{T} \pi_1 R_\lambda(u, v) = \lambda u - \mathrm{i} v + \lambda^2 \pi_1 R_\lambda(u, v) \ \in L^2_w$
    $\implies \pi_1 R_\lambda(u, v) \in \mathrm{Dom}(T)$
  - $\pi_2 R_\lambda(u, v) = \mathrm{i}(\hat{T} - \lambda^2)^{-1} \hat{T} u - \lambda(\hat{T} - \lambda^2)^{-1} v \ \in \mathcal{W}$

- $(u,v) \in \mathcal{W} \oplus L^2_w,\ \ R_\lambda(u,v) \in \mathrm{Dom}(L)$:
  - $\pi_1 R_\lambda(u,v) = -\frac{1}{\lambda} u + \frac{1}{\lambda}(\hat{T} - \lambda^2)^{-1} \hat{T} u - \mathrm{i}(\hat{T} - \lambda^2)^{-1} v \ \in \mathcal{W}$
    $\hat{T} \pi_1 R_\lambda(u,v) = \lambda u - \mathrm{i} v + \lambda^2 \pi_1 R_\lambda(u,v) \ \in L^2_w$
    $\implies \ \pi_1 R_\lambda(u,v) \in \mathrm{Dom}(T)$
  - $\pi_2 R_\lambda(u,v) = \mathrm{i}(\hat{T} - \lambda^2)^{-1} \hat{T} u - \lambda(\hat{T} - \lambda^2)^{-1} v \ \in \mathcal{W}$
- $\forall (u,v) \in \mathcal{W} \oplus L^2_w :\ (L - \lambda) R_\lambda(u,v) = (u,v)$
- $\forall (u,v) \in \mathrm{Dom}(L) :\ R_\lambda (L - \lambda)(u,v) = (u,v)$.

$$\Rightarrow R_\lambda = (L - \lambda)^{-1} \Rightarrow \boxed{(L - \lambda)^{-1} \in \mathscr{B}(\mathcal{W} \oplus L^2_w)}$$

□

Pseudospectrum [2]

- ε-pseudospectrum of L:

$$\sigma_\varepsilon(L) := \sigma(L) \cup \{z \in \mathbb{C} \ : \ \|(L-z)^{-1}\| > \varepsilon^{-1}\}$$

- Relation to spectrum:

$$\|(L-z)^{-1}\| \geq \operatorname{dist}(z, \sigma(L))^{-1},$$
$$\{z \in \mathbb{C} \ : \ \operatorname{dist}(z, \sigma(L)) < \varepsilon\} \subseteq \sigma_\varepsilon(L)$$

- Spectral instability:

$$\sigma_\varepsilon(L) = \bigcup_{\|V\|<\varepsilon} \sigma(L+V)$$

Pseudospectrum and black holes

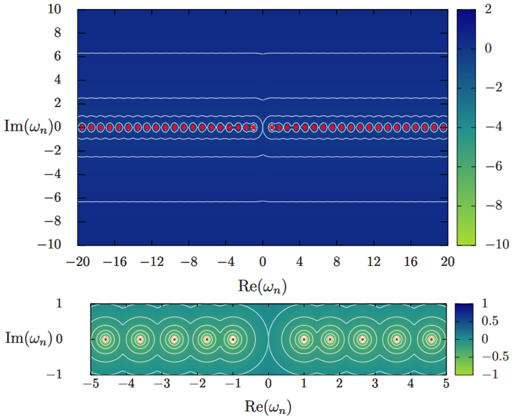

Figure: Pseudospectrum of a self-adjoint operator with a toy model potential [1]

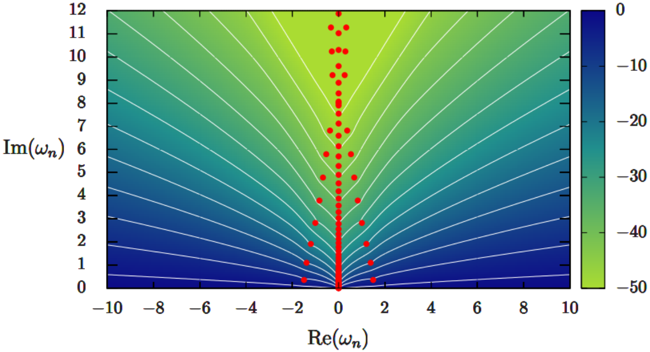

Figure: Pseudospectrum of the non-self-adjoint Schwarzschild case [1]

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