Discrete Dirac operator and its non-relativistic limit

Ruben Karapetyan

Supervisor: Ing. Matěj Tušek, Ph.D.

FNSPE CTU in Prague

Outline

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Motivation

Physics

1D example in space $L^2(\mathbb{R})$:
$\hat{H} = -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2} + \hat{V} \approx -\triangle + \hat{V}$

Discrete 1D version

Space $\ell^2(\mathbb{Z})$ of sequences $(a_n)_{n=-\infty}^{+\infty}$ where $n \in \mathbb{Z}$

$$\hat{H} = H + V \quad \text{with } V \in B(\ell^2),\ H^\varepsilon = \frac{1}{2m}\begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \\ \ddots & 2 & -1 & 0 & \ddots \\ \ddots & -1 & 2 & -1 & \ddots \\ \ddots & 0 & -1 & 2 & \ddots \\ & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Necessary definitions

Operators on $B(\ell^2(\mathbb{Z}))$

$e_n = (\dots,\ 0,\ 1,\ 0,\dots) \Longrightarrow \varepsilon = (e_n)_{n\in\mathbb{Z}}$ standard ON basis
$d^- e_n = -i(e_n - e_{n-1})$ and $d^+ e_n = -i(e_{n+1} - e_n)$,
then $H = \frac{1}{2m} d^+ d^-$

Operators on $B(\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z}))$

$H \oplus 0 = \begin{pmatrix} \frac{1}{2m} d^+ d^- & 0 \\ 0 & 0 \end{pmatrix}$, H non-relativistic Laplacian

$D_c = \begin{pmatrix} mc^2 & cd^- \\ cd^+ & -mc^2 \end{pmatrix}$ discrete relativistic Dirac operator

Resolvent

$R_{\mathbb{A}}(\lambda) = (\mathbb{A} - \lambda \mathbb{I})^{-1}$ for $\lambda \in \rho(\mathbb{A})$

Dirac operator

- is formally a square root of a Laplacaian
- discrete Dirac

$$D_c^2 = \begin{pmatrix} m^2c^4 - c^2 d^- d^+ & 0 \\ 0 & -m^2c^4 + c^2 d^+ d^- \end{pmatrix}$$
$$= \begin{pmatrix} m^2c^4 - 2mc^2 H & 0 \\ 0 & -m^2c^4 + 2mc^2 H \end{pmatrix}$$

- We study shifted discrete Dirac

$$D_c - mc^2\mathbb{I} = \begin{pmatrix} 0 & cd^- \\ cd^+ & -2mc^2 \end{pmatrix}$$

Goals of the thesis

- Convergence of shifted Dirac in norm resolvent sense

$$\left\|(D_c - mc^2 - \lambda)^{-1} - (H - \lambda)^{-1} \oplus 0\right\|_{B(\ell^2 \oplus \ell^2)} \xrightarrow{c \to \infty} 0$$

- Taylor series of $(D_c - mc^2 - \lambda)^{-1}$
- Adding potential $V \in B(\ell^2 \oplus \ell^2)$

$$\lim_{c \to \infty} \left(D_c - mc^2 + V - \lambda\right)^{-1} = \text{? in } B(\ell^2 \oplus \ell^2)$$

Laurent operators

Principle

Using U we map the ON basis of $B(\ell^2)$ to $B(L^2)$.
By this we transform our problem from $B(\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z}))$ to $B\left(L^2([0,2\pi]) \oplus L^2([0,2\pi])\right)$.
U is a linear bijective isometry.

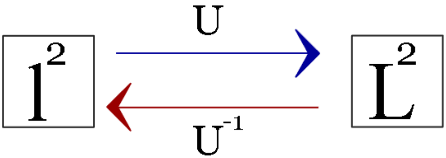

Figure: principle of using Laurent operators

Laurent operators

Requirement of constant diagonals.
E.g.

$$d^+ e_n = -i(e_{n+1} - e_n),$$

$$(d^+)^\varepsilon = -i \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & -1 & 1 & 0 & \ddots \\ \ddots & 0 & -1 & 1 & \ddots \\ \ddots & 0 & 0 & -1 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Laurent operators

Transfering the operators

1. $U(\pm mc^2 \mathbb{I}_{\ell_2})U^{-1} = \pm mc^2 \mathbb{I}_{L_2}$
2. $Ud^+U^{-1} = \big(-i(e^{it} - 1)\big)$
3. $Ud^-U^{-1} = \big(-i(1 - e^{-it})\big)$
4. $UHU^{-1} = \big(\frac{1-\cos t}{m}\big)$
5. $U \oplus U(D_c - mc^2 \mathbb{I}_{\ell_2})U^{-1} \oplus U^{-1} =$
   $= \begin{pmatrix} U0U^{-1} & Ucd^+U^{-1} \\ Ucd^-U^{-1} & U(-2mc^2)U^{-1} \end{pmatrix} =$
   $\begin{pmatrix} 0 & -ic(e^{it} - 1) \\ -ic(1 - e^{-it}) & -2mc^2 \mathbb{I}_{L_2} \end{pmatrix} = \mathbb{D}_c$
6. $U \oplus U \ (H \oplus 0) \ U^{-1} \oplus U^{-1} = \begin{pmatrix} \frac{1-\cos t}{m} & 0 \\ 0 & 0 \end{pmatrix} = \mathbb{H} \oplus 0$

Resolvent in $B(L^2 \oplus L^2)$

Formulation of the problem in $B(L^2 \oplus L^2)$

Goal: convergence in norm resolvent sense of $\mathbb{D}_c$

$$(\mathbb{D}_c - \lambda)^{-1} = \begin{pmatrix} \frac{-2mc^2-\lambda}{\lambda^2+2\lambda mc^2+2c^2(\cos t-1)} & \frac{ic(1-e^{-it})}{\lambda^2+2\lambda mc^2+2c^2(\cos t-1)} \\ \frac{ic(e^{it}-1)}{\lambda^2+2\lambda mc^2+2c^2(\cos t-1)} & \frac{-\lambda}{\lambda^2+2\lambda mc^2+2c^2(\cos t-1)} \end{pmatrix}$$

to the operator

$$(\mathbb{H}-\lambda)^{-1} \oplus 0 = \begin{pmatrix} (\frac{1-\cos t}{m} - \lambda)^{-1} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{m}{1-\cos t - m\lambda} & 0 \\ 0 & 0 \end{pmatrix}.$$

Laurent operators

Theory

1. $A, B, C, D \in B(L^2([0, 2\pi]))$

$$\left\| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\|_{B(L^2 \oplus L^2)} \quad (1)$$

$$\leq K \left(\|A\|_{B(L^2)} + \|B\|_{B(L^2)} + \|C\|_{B(L^2)} + \|D\|_{B(L^2)} \right) \quad (2)$$

for some $K > 0$.

2. $f \in L^2([0, 2\pi])$

$$\|M_f\|_{B(L^2)} = \|f\|_\infty = inf\{c > 0 | \ \mu(\{t \in [0, 2\pi] | f(t) > c\}) = 0\}. \quad (3)$$

Laurent operators

Using (1), (3)

$$
\begin{aligned}
&\left\|(\mathbb{D}_c-\lambda)^1-(\mathbb{H}-\lambda)^{-1}\oplus 0\right\|_{B(L^2\oplus L^2)}= \\
&=\left\|\begin{pmatrix}\frac{-2mc^2-\lambda}{\lambda^2+2\lambda mc^2+2c^2(\cos t-1)} & \frac{ic(1-e^{-it})}{\lambda^2+2\lambda mc^2+2c^2(\cos t-1)}\\ \frac{ic(e^{it}-1)}{\lambda^2+2\lambda mc^2+2c^2(\cos t-1)} & \frac{-\lambda}{\lambda^2+2\lambda mc^2+2c^2(\cos t-1)}\end{pmatrix}-\begin{pmatrix}\frac{m}{1-\cos t-m\lambda} & 0\\ 0 & 0\end{pmatrix}\right\|_{B(L^2\oplus L^2)} \\
&\leq\left\|\frac{-2mc^2-\lambda}{\lambda^2+2\lambda mc^2+2c^2(\cos t-1)}-\frac{m}{1-\cos t-m\lambda}\right\|_{B(L^2)}+\left\|\frac{ic(1-e^{-it})}{\lambda^2+2\lambda mc^2+2c^2(\cos t-1)}\right\|_{B(L^2)} \\
&+\left\|\frac{ic(e^{it}-1)}{\lambda^2+2\lambda mc^2+2c^2(\cos t-1)}\right\|_{B(L^2)}+\left\|\frac{-\lambda}{\lambda^2+2\lambda mc^2+2c^2(\cos t-1)}\right\|_{B(L^2)} \\
&\leq\left|\frac{2\lambda}{KLc^2}\right|+\left|\frac{2}{Lc}\right|+\left|\frac{2}{Lc}\right|+\left|\frac{\lambda}{Lc^2}\right|\xrightarrow{c\to\infty}0.
\end{aligned}
$$

Laurent operators

Taylor's series in $1/c$

$$(\mathbb{D}_c - \lambda)^{-1} = (\mathbb{H} - \lambda)^{-1} \oplus 0 + \frac{1}{c}\mathbb{X}_1 + \frac{1}{c^2}\mathbb{X}_2 + \frac{1}{c^3}\mathbb{X}_3 + \dots$$

meaning

$$\left\|(\mathbb{D}_c - \lambda)^{-1} - (\mathbb{H} - \lambda)^{-1} \oplus 0 - \textstyle\sum_{n=1}^{\infty} \frac{1}{c^n} X_n\right\|_{B(L^2 \oplus L^2)} = 0$$

for some neighborhood of $1/c = 0$.

Laurent operators

Candidates using classical Taylor's expansion in $1/c$

$$\mathbb{X}_{2j} =$$

$$\begin{pmatrix} \left(\frac{-2m}{2(\lambda m+\cos t-1)} + \frac{1}{\lambda}\right)\left(\frac{-\lambda^2}{2(\lambda m+\cos t-1)}\right)^j & 0 \\ 0 & \frac{-\lambda}{2(\lambda m+\cos t-1)}\left(\frac{-\lambda^2}{2(\lambda m+\cos t-1)}\right)^{j-1} \end{pmatrix}$$

$$\mathbb{X}_{2j+1} =$$

$$\begin{pmatrix} 0 & \frac{i(1-e^{-it})}{(\lambda m+\cos t-1)}\left(\frac{-\lambda^2}{2(\lambda m+\cos t-1)}\right)^j \\ \frac{i(e^{it}-1)}{(\lambda m+\cos t-1)}\left(\frac{-\lambda^2}{2(\lambda m+\cos t-1)}\right)^j & 0 \end{pmatrix}.$$

Laurent operators

Verifying the candidates are correct

Theorem

Suppose $f_\varepsilon(t)$ function with parametre $\varepsilon \in [0, M)$, for some $M > 0$ is $(\forall \varepsilon < M)\left(f_\varepsilon(\cdot) \in L^2([0, 2\pi])\right)$.
Suppose we have full Taylor expansion for $\varepsilon = 0$

$$\forall t \in [0, 2\pi] f_\varepsilon(t) = \sum_{n=0}^{\infty} a_n(t)\varepsilon^n = a_0(t) + \sum_{n=1}^{\infty} a_n(t)\varepsilon^n,$$

$\exists G, L > 0 \ \left\|a_n\right\|_\infty \leq GL^n$.
If $M_f \in B\left(L^2([0, 2\pi])\right)$ is multiplication operator, $\forall N \in \mathbb{N}_0$ $f_\varepsilon^N(t) = a_0 + \sum_{n=1}^N a_n\varepsilon^n$, then

$$\left\|M_f - M_{f_\varepsilon^N}\right\|_{B(L^2([0,2\pi]))} \leq GL^{N+1}\varepsilon^{N+1} \overset{\varepsilon \to 0+}{\longrightarrow} 0.$$

Laurent operators

We obtain for some neighborhood of $c = +\infty$

$$\left(D_c - mc^2 - \lambda\right)^{-1} = (H - \lambda)^{-1} \oplus 0 + \sum_{n=1}^{\infty} \frac{1}{c^n} X_n \text{ in } B(\ell^2 \oplus \ell^2)$$

by transforming

$$X_n = U^{-1} \oplus U^{-1} \mathbb{X}_n U \oplus U.$$

$$X_{2j} =$$

$$\begin{pmatrix} \left(\frac{1}{\lambda} + (H-\lambda)^{-1}\right)\left(\frac{\lambda^2}{2m}(H-\lambda)^{-1}\right)^j & 0 \\ 0 & \frac{\lambda}{2m}(H-\lambda)^{-1}\left(\frac{\lambda^2}{2m}(H-\lambda)^{-1}\right)^j \end{pmatrix}.$$

$$X_{2j+1} =$$

$$\begin{pmatrix} 0 & -d^- \frac{1}{m}(H-\lambda)^{-1}\left(\frac{\lambda^2}{2m}(H-\lambda)^{-1}\right)^j \\ -d^+ \frac{1}{m}(H-\lambda)^{-1}\left(\frac{\lambda^2}{2m}(H-\lambda)^{-1}\right)^j & 0 \end{pmatrix}.$$

Matrix decomposition

Ch. Tretter - Spectral Theory of Block Operator Matrices

Let $A, B, C, D \in B(\mathscr{H})$,

$$\mathbb{X} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in B(\mathscr{H} \oplus \mathscr{H}).$$

Then for $\lambda \notin \sigma(D)$

$$\mathbb{X} - \lambda = \begin{pmatrix} \mathbb{I} & B(D-\lambda)^{-1} \\ 0 & \mathbb{I} \end{pmatrix} \begin{pmatrix} S(\lambda) & 0 \\ 0 & D-\lambda \end{pmatrix} \begin{pmatrix} \mathbb{I} & 0 \\ (A-\lambda)^{-1}B & \mathbb{I} \end{pmatrix},$$

where $S(\lambda) = A - \lambda - B(D-\lambda)^{-1}C$ is the Schur complement of $\mathbb{X}$.

Matrix decomposition

Using the decomposition

In our case $\mathscr{H} = \ell^2$.

$$D_c - mc^2 - \lambda =$$
$$= \begin{pmatrix} \mathbb{I} & cd^-(-2mc^2-\lambda)^{-1} \\ 0 & \mathbb{I} \end{pmatrix} \begin{pmatrix} S(\lambda) & 0 \\ 0 & -2mc^2-\lambda \end{pmatrix} \begin{pmatrix} \mathbb{I} & 0 \\ (-2mc^2-\lambda)^{-1}cd^- & \mathbb{I} \end{pmatrix}$$

where

$$S(\lambda) = cd^-(2mc^2+\lambda)^{-1}cd^+ - \lambda.$$

Matrix decomposition

Resolvent

For bounded operators, $\lambda \in \rho\left(D_c - mc^2\right)$

$$(D_c - mc^2\mathbb{I} - \lambda)^{-1} =$$
$$\begin{pmatrix} \mathbb{I} & 0 \\ -(-2mc^2-\lambda)^{-1}cd^- & \mathbb{I} \end{pmatrix} \begin{pmatrix} S^{-1}(\lambda) & 0 \\ 0 & \frac{-1}{2mc^2+\lambda} \end{pmatrix} \begin{pmatrix} \mathbb{I} & -cd^-(-2mc^2-\lambda)^{-1} \\ 0 & \mathbb{I} \end{pmatrix}.$$

Theory

Multiplying bounded operators is continuous.
For $(X_n)_{n=1}^{+\infty}, (Y_n)_{n=1}^{+\infty}, (Z_n)_{n=1}^{+\infty} \subset B(\mathscr{H})$:

$$X_n \xrightarrow{n\to\infty} X,\ Y_n \xrightarrow{n\to\infty} Y,\ Z_n \xrightarrow{n\to\infty} Z \implies X_nY_nZ_n \xrightarrow{n\to\infty} XYZ$$

in the strong operator topology.

Matrix decomposition

Proving that

$$\begin{pmatrix} \mathbb{I} & 0 \\ -(-2mc^2-\lambda)^{-1}cd^- & \mathbb{I} \end{pmatrix} \xrightarrow{c\to\infty} \begin{pmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix}$$

$$\begin{pmatrix} S^{-1}(\lambda) & 0 \\ 0 & \frac{-1}{2mc^2+\lambda} \end{pmatrix} \xrightarrow{c\to\infty} \begin{pmatrix} (H-\lambda)^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \mathbb{I} & -cd^-(-2mc^2-\lambda)^{-1} \\ 0 & \mathbb{I} \end{pmatrix} \xrightarrow{c\to\infty} \begin{pmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix}$$

we obtain for $\forall\lambda \in \rho\left(D_c - mc^2\mathbb{I}\right)$

$$(D_c - mc^2 - \lambda)^{-1} \xrightarrow{c\to\infty} (H-\lambda)^{-1} \oplus 0 \;\text{ in }\; B(\ell^2 \oplus \ell^2).$$

Matrix decomposition

Adding potential

Let $V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \in B(\ell^2 \oplus \ell^2)$, $\lambda \in \rho\left(D_c - mc^2\mathbb{I} + V\right)$

$$\begin{aligned}
&\left(D_c - mc^2 + V - \lambda\right)^{-1} = \\
&= \begin{pmatrix} \mathbb{I} & 0 \\ -\left((-2mc^2 - \lambda)\mathbb{I} + V_{22}\right)^{-1}\left(cd^+ + V_{21}\right) & \mathbb{I} \end{pmatrix} \times \\
&\times \begin{pmatrix} \hat{S}_1^{-1}(\lambda) & 0 \\ 0 & \left((-2mc^2 - \lambda)\mathbb{I} + V_{22}\right)^{-1} \end{pmatrix} \times \\
&\times \begin{pmatrix} \mathbb{I} & -\left(cd^- + V_{12}\right)\left((-2mc^2 - \lambda)\mathbb{I} + V_{22}\right)^{-1} \\ 0 & \mathbb{I} \end{pmatrix}.
\end{aligned}$$

Matrix decomposition

Proving that

$$\begin{pmatrix} \mathbb{I} & 0 \\ -\left((-2mc^2-\lambda)+V_{22}\right)^{-1}(cd^+ + V_{21}) & \mathbb{I} \end{pmatrix} \xrightarrow{c\to\infty} \begin{pmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix}$$

$$\begin{pmatrix} \hat{S}_1^{-1}(\lambda) & 0 \\ 0 & \left((-2mc^2-\lambda)+V_{22}\right)^{-1} \end{pmatrix} \xrightarrow{c\to\infty} \begin{pmatrix} (H+V_{11}-\lambda)^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \mathbb{I} & -(cd^- + V_{12})\left((-2mc^2-\lambda)+V_{22}\right)^{-1} \\ 0 & \mathbb{I} \end{pmatrix} \xrightarrow{c\to\infty} \begin{pmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix}$$

we obtain

$$(D_c - mc^2 + V - \lambda)^{-1} \xrightarrow{c\to\infty} (H+V_{11}-\lambda)^{-1} \oplus 0 \;\text{ in }\; B(\ell^2 \oplus \ell^2).$$

Supersymetry

Bernd Thaller - The Dirac Equation, 1992

$D \in \mathscr{L}(\mathscr{H})$ is a Dirac operator with supersymetry if

$$D = Q + M\tau,$$

where

- Q is self-adjoint and $\tau Q = -Q\tau$ (Q is a supercharge with respect to τ)
- $M > 0$ commutes with Q, τ

$$D_c = \underbrace{c\begin{pmatrix} 0 & d^- \\ d^+ & 0 \end{pmatrix}}_{Q} + \underbrace{mc^2\mathbb{I}}_{M}\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\tau}$$

$$D_\infty = \frac{1}{2m}Q^2 = \frac{1}{2m}\begin{pmatrix} d^-d^+ & 0 \\ 0 & d^+d^- \end{pmatrix} = \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix} = H \oplus H.$$

Supersymetry

Bernd Thaller - The Dirac Equation (1992)

For $\forall \lambda \in \mathbb{C} \setminus \mathbb{R}$

$$(D_c - mc^2 - \lambda)^{-1} =$$

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2mc^2}(cQ + \lambda) \right) \left(\mathbb{I} - \frac{\lambda^2}{2mc^2}(H \oplus H - \lambda)^{-1} \right)^{-1} (H \oplus H - \lambda)^{-1}$$

$$\xrightarrow{c \to \infty} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (H - \lambda \mathbb{I})^{-1} & 0 \\ 0 & (H - \lambda \mathbb{I})^{-1} \end{pmatrix}.$$

Supersymetry

Adding potential, analyticity

Using Theorem 6.4, 6.5 (B. Thaller, 1992)

Let $V = \begin{pmatrix} V_{11} & 0 \\ 0 & V_{22} \end{pmatrix} \in B(\ell^2 \oplus \ell^2)$ be a symmetric potential.

Then $\left(D_c - mc^2\mathbb{I} + V - \lambda\right)^{-1}$ is holomorphic in $1/c$ on a λ depending neighborhood of $1/c = 0$.

$$\left(D_c - mc^2 + V - \lambda\right)^{-1} = \sum_{n=0}^{+\infty} \frac{1}{c^n} R_n(\lambda),$$

where

$$R_0(\lambda) = \begin{pmatrix} (H + V_{11} - \lambda)^{-1} & 0 \\ 0 & 0 \end{pmatrix},$$

$$R_1(\lambda) = \begin{pmatrix} 0 & \frac{1}{m} d^+ (H + V_{11} - \lambda)^{-1} \\ \frac{1}{m}(H + V_{11} - \lambda)^{-1} d^- & 0 \end{pmatrix}.$$

Thank you for attention.

References

- Christiane Tretter (2008)
  Spectral Theory Of Block Operator Matrices And Applications
- Bernd Thaller (1992)
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