Magnetic Effects in the Spectrum of Laterally Coupled Layers

Master Thesis

Bc. Ondřej Šrámek
Supervisor: prof. RNDr. Pavel Exner, DrSc.

Faculty of Nuclear Science and Physical Engeneering

9. May 2023

Contents

Absolute Continuous Spectrum

- Decomposition of measure $\mu = \mu_{ac} + \mu_{sc} + \mu_{pp}$

Definition

Measure μ is absolutely continuous with respect to Lebesgue measure λ if and only if $\forall A,\ \lambda(A) = 0 \Rightarrow \mu(A) = 0$.

- When μ is spectral measure of operator H then

$$\sigma_{ac}(H) = \operatorname{supp}(\mu_{ac}),$$
$$\sigma_{sc}(H) = \operatorname{supp}(\mu_{sc}),$$
$$\sigma_{p}(H) = \operatorname{supp}(\mu_{pp}).$$

[8]

Direct Integral

- The concept of a elegant solution to spectral properties
- Separable Hilbert space $\mathscr{H}'$ with measure (M, μ)
- $\mathscr{H} = L^2(M, d\mu; \mathscr{H}')$ space of all square integrable function from M to $\mathscr{H}'$
- $\mathscr{H}$ is called constant fiber direct integral and we write $\mathscr{H} = \int_{\oplus M} \mathscr{H}' d\mu$
- "Continuous generalization of direct sum"[1]

Definition

A bounded operator A on $\int_{\oplus M} \mathscr{H}$ is said to be decomposed by the direct integral decomposition if and only if there is a function $A(\cdot)$ in $L^{\infty}(M, d\mu, \mathscr{L}(\mathscr{H}'))$ so that for all $\psi \in \mathscr{H}$,

$$(A\psi)(\xi) = A(\xi)\psi(\xi)$$

We then call A decomposable and write

$$A = \int_{\oplus M} A(\xi)\, d\mu(\xi)$$

The $A(\xi)$ are called the fibers of A.[1]

- Definition for an unbounded operator is a little bit different
- Foreshorten the operator domain in unbounded case

$$D(A) = \{\psi \in \mathscr{H} \mid \psi(\xi) \in D(A(\xi)) \text{ a.e.;.} \\ \int_M \| A(\xi)\psi(\xi) \|^2_{\mathscr{H}'} d\mu(\xi) < \infty\}$$

[1]

Iwatsuka Model

- $H = \left(\frac{1}{i}\frac{\partial}{\partial x} - a(x,y)\right)^2 + \left(\frac{1}{i}\frac{\partial}{\partial y} - b(x,y)\right)^2$ on $\mathscr{H} = L^2(\mathbb{R}^2)$,where $a, b \in C^\infty(\mathbb{R}^2)$
$$B(x,y) = \frac{\partial b}{\partial x}(x,y) - \frac{\partial a}{\partial y}(x,y)$$
- The asymptotically constant B, it means $B(x,y) \overset{\sqrt{x^2+y^2}\to\infty}{-\!-\!\longrightarrow} B_0$.
  - For $B = 0$, $\sigma_{ess}(H) = [0, \infty)$
  - Otherwise
$$\sigma_{ess}(H) = \{(2k-1)|B_0| \mid k \in \mathbb{N}\}.$$
    [2]

Iwatsuka Model

Theorem

$B(x,y)$ depends only on x, $B(x) \in C^\infty$ and there exists constant $M_\pm$ satisfying $0 < M_- \le B(x) \le M_+ < +\infty$ for all x

1. $\limsup_{x\to-\infty} B(x) < \liminf_{x\to+\infty} B(x)$ *or* $\limsup_{x\to+\infty} B(x) < \liminf_{x\to-\infty} B(x)$
2. *$B(x) = B_0$ if $|x|$ is large and there is a point $\bar{x}$ such that $B'(x) \le 0$ for $x \le \bar{x}$ and $B'(x) \ge 0$ for $x \ge \bar{x}$ and $B'(x)$ is not identically 0.*

If holds 1. or 2. Then spectrum of H is Absolutely continuous.

- If $\lim_{x\to\pm\infty} B(x) = B_\pm$ then spectrum has a band structure

$$\sigma(H) = \bigcup_{n=1}^{\infty} \left[(2n-1)B_-, (2n-1)B_+\right]$$

Iwatsuka Model

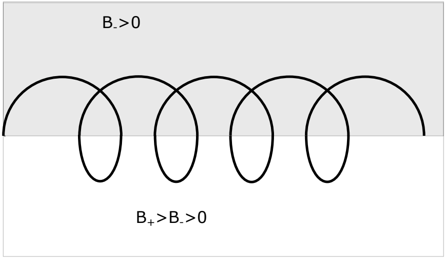
- Alternative theorem [3]

Theorem

Let holds previous property and in addition B is not constant a there is a point $x_0 \in \mathbb{R}$ such that for all points $x_1 \leq x_0 \leq x_2$ holds one of the following conditions

1. $B(x_1) \leq B(x_0) \leq B(x_2)$
2. $B(x_1) \geq B(x_0) \geq B(x_2)$

Then H is absolutely continuous.



Obrázek: Motion of electron

Effect of Potential

- Potential wall

$$U(x) = 0 \quad \text{for } x \leq 0, \qquad (1)$$
$$U(x) = \mu x^{\gamma} \quad \text{for } x > 0,$$

$\mu > 0, \ \gamma \geq 1.$

$$H_0 = H + U(x) \Rightarrow \sigma(H_0) = [B, +\infty]. \qquad (2)$$

- Potential on the whole plane $U(x) = \mu x \ \forall x \Rightarrow \sigma(H_0) = \mathbb{R}$
- Adding a random potential

$$V_{\omega}(\vec{r}) = \sum_{(n,m)\in\mathbb{Z}} \omega_{n,m} v(x-n, y-m), \qquad (3)$$

where $v(\vec{r}) = 0$ for $|\vec{r}| \geq \frac{1}{2}$ and $\omega_{n,m}$ is i.i.d. random variable, the discrete spectrum can be found [4]

Introduction to Problem

- Lets have a problem with the magnetic Laplacian

$$H\psi(r,\theta,z) = \left(i\nabla + \vec{A}\right)^2 \psi(r,\theta,z) \tag{4}$$

with Dirichlet condition

$$\psi(r,\theta,d_1) = 0,$$
$$\psi(r,\theta,0) = 0 \text{ pro } (r,\theta,0) \notin \Omega_0,$$
$$\psi(r,\theta,-d_2) = 0.$$

- So the domain of operator

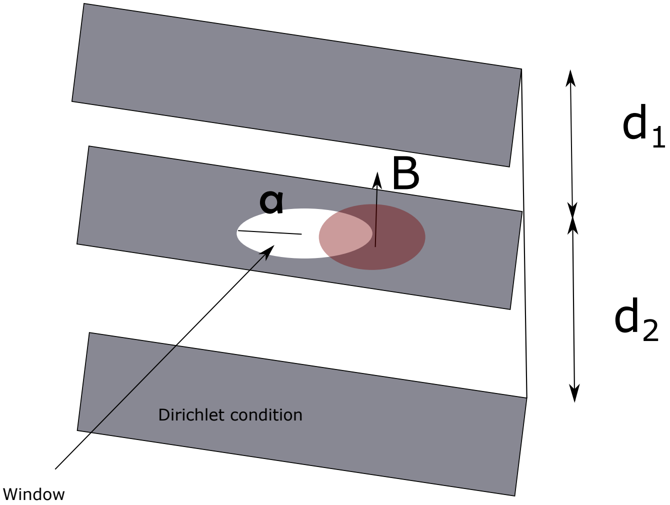
$$D(H) = \{\psi \in H^2(\Omega) \mid \left(i\nabla + \vec{A}\right)^2 \psi \in L^2(\Omega)$$
$$\psi(\vec{x}) = 0 \text{ pro } \vec{x} \in \partial\Omega \cup \Sigma \setminus \Omega_0\}$$

Local Magnetic Field

- Definition of local magnetic field [7]
  Choose $p \in \Omega_B$ so that $\exists R > 0, B(p, R) \subset \Omega_B$. For $r \in (0, R)$
  $$\Phi(r) = \frac{1}{2\pi} \int_{B(r,p)} B dx dy dz$$
  is not identically equal zero.
- In area Ω_B, the field is homogeneous and vector potential is given as $\vec{A} = \left(-\frac{1}{2} B(y - y_0), \frac{1}{2} B(x - x_0), 0\right)$
- Outside of area Ω_B, the field is zero and vector potential is

$$\vec{A} = \Phi \left(\frac{-y + y_0}{(x - x_0)^2 + (y - y_0)^2}, \frac{x - x_0}{(x - x_0)^2 + (y - y_0)^2}, 0 \right). \quad (5)$$

Laterally coupled layers with window and area with non-zero magnetic field



Symmetric Case

- For $d_1 = d_2 =: d$ it is enough to consider only one layer with Neumann condition in window

Essential Spectrum

Theorem

$\lambda \in \sigma_{ess}(H) \iff$ *Exist* $(\psi_n) \subset D(H)$, *for every* $n \in \mathbb{N}$, $\|\psi_n\| = 1$ *a* $T\psi_n - \lambda\psi_n \to 0, \psi_n \xrightarrow{w} 0$

Theorem

Spectrum of H^0 *and* $H^{+\infty}$ *holds* $\sigma(H^0) = [\left(\frac{\pi}{d}\right)^2, +\infty)$ *and* $\sigma(H^{+\infty}) = [\left(\frac{\pi}{2d}\right)^2, +\infty)$.
And from that comes H

Theorem

For essential spectrum of H holds

$$\sigma_{ess}(H) = [\left(\frac{\pi}{d}\right)^2, +\infty).$$

Discrete Spectrum

- Neumann-Dirichlet bracketing

$$\Omega_\varrho^- = \{(r,\theta,z) \in [0,\varrho] \times [0,2\pi] \times [0,d]\},$$
$$\Omega_\varrho^+ = \Omega \diagdown \Omega_\varrho^-$$

- Application of Dirichlet-Neumann bracketing

$$H_\varrho^{-,N} \oplus H_\varrho^{+,N} \leq H \leq H_\varrho^{-,D} \oplus H_\varrho^{+,D}$$

- Reduction problem to plane
- Use of Laplacian monotonicity

Theorem

[1] Let Ω_1 and Ω_2 are bounded areas. In addition, $\Omega_1 \subset \Omega_2$. Then for eigenvalues of Dirichlet magnetic laplacian $\left(i\nabla + \vec{A}\right)^2$ it holds $\lambda_k(\Omega_2) \leq \lambda_k(\Omega_1)$ for all k.

- Without prejudice to the generality reduction to disc
- Conditions of existence of eigenstates are given from function

$$\Psi(r) = c \cdot e^{-\frac{Br^2}{4}} r^m M\left(\frac{k^2 - \lambda + B}{2B}, m+1, \frac{Br^2}{2}\right)$$
$$\Psi(r) = cJ_{|m-\Phi|}\left(\sqrt{\lambda} r\right)$$

Dirichlet Bracketing

$$M\left(-\frac{\lambda_D^2 - B}{2B}, 1, \frac{1}{2}B\varrho^2\right) = 0 \qquad J_{|m-\Phi|}\left(\sqrt{\lambda_D}\varrho\right) = 0$$

Theorem

Let $B\varrho^2 \geq 4$, then operator $H = \left(i\nabla + \vec{A}\right)^2$ defined in equation (4) has nonempty discrete spectrum, If there is a disc with radius ϱ inside $\Omega_0 \cap \Omega_B$ such that

$$B + eB^2\varrho^2 e^{-\frac{1}{2}B\varrho^2} < \frac{3}{4}\frac{\pi^2}{d^2}.$$

Theorem

Magnetic laplacian H has nonempty discrete spectrum, if there is a disc with radius $\varrho > 0$ inside $\Omega_0 \setminus \Omega_B$ such that holds

$$\frac{2}{\sqrt{3}\pi}\sqrt{\left(\frac{3\pi}{4}\right)^2 + \Phi^2} < \frac{\varrho}{d}.$$

Neumann Bracketing

$$-M\left(-\frac{\lambda_N^2 - B}{2B}, 1, \frac{1}{2}Ba^2\right) + \frac{-\lambda_N^2 + B}{B} M\left(-\frac{\lambda_N^2 - 3B}{2B}, 2, \frac{1}{2}Ba^2\right) = 0,$$

$$\frac{d}{dr} J_{|m-\Phi|}\left(\sqrt{\lambda_D}\varrho\right) = 0$$

Theorem

Operator $\left(i\nabla + \vec{A}\right)^2$ defined in equation (4) has empty discrete spectrum, if there exists disc $D(a, \varrho) \supset \Omega_0 \cap \Omega_B$ with radius $\varrho > 0$ a $C > 0$, $B\varrho^2 \geq C$ such that

$$\Theta_0 B - C_1 \frac{1}{\varrho} B^{\frac{1}{2}} - C\frac{1}{\varrho^2} > \frac{3}{4}\left(\frac{\pi}{d}\right)^2.$$

Theorem

Magnetic laplacian (4) has empty discrete spectrum, if there is disc $D(a, \varrho) \supset \Omega_0 \setminus \Omega_B$ with radius $\varrho > 0$, such that

$$\frac{2}{\sqrt{3}\pi}\sqrt{0.6538 + \Phi} > \frac{\varrho}{d}.$$

Numerical results

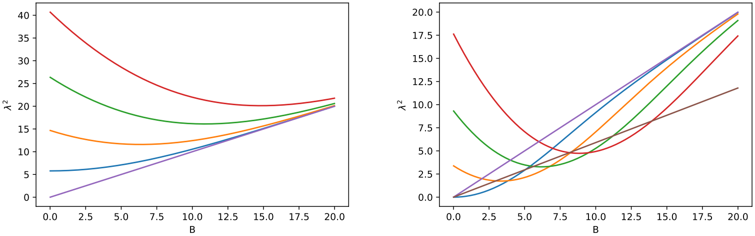


Figure: Dirichlet and Neumann condition for Kummer function. Blue color denotes numerical solution λ_D^2, λ_N^2 with respect to B for $m = 0$, $\varrho = 1$. Orange $m = 1$, green $m = 2$, red $m = 3$. Brown curve denotes function $\Theta_0 B$ and purple B.

Asymmetric Case

- $d := \max\{d_1, d_2\}$
- Total width of both layers $D := d_1 + d_2$
- Ratio $\eta = \frac{\min\{d_1, d_2\}}{\max\{d_1, d_2\}}$.

Essential Spectrum

$\sigma_{ess}(H) = [\left(\frac{\pi}{d}\right)^2, +\infty)$

Theorem

Let H is operator defined in (4). Then H has discrete spectrum if and only if the lowest eigenvalue λ_D^2 of magnetic laplacian $\left(i\nabla + \hat{\vec{A}}\right)_D^2$ in area $\Omega_0 \setminus \Omega_B$ resp. $\Omega_0 \cap \Omega_B$ with Dirichlet condition holds

$$\lambda_{\mathrm{D}}^2 < \left(\frac{\pi}{\mathrm{d}}\right)^2 \frac{(\eta^2 + 2\eta)}{(1+\eta)^2}$$

Homogeneous Magnetic Field in Whole Layer

- Direct integral $H \simeq \int_{\oplus} H(\xi)\, d\xi$
- $H(\xi) = -\frac{\partial^2}{\partial x^2} + (\xi + Bx)^2 - \frac{\partial}{\partial z^2}$
- Separation of variables
- Eigenvalues of $H(\xi)$ are non-degenerate $\lambda_{n,m} = B(2n+1) + \left(\frac{\pi m}{d}\right)^2$
- Spectrum has pure point structure with eigenfunctions

$$\psi_{n,m}(x,z) = C \cdot X_n\left(x + \frac{\xi}{B}\right) \sin\left(\frac{\pi m z}{d}\right),.$$

$$X_n(\tau) = \left(\frac{B}{\pi}\right)^{\frac{1}{4}} e^{-B\tau^2} H_n\left(\sqrt{B}\tau\right).$$

- Then spectrum of H is essential with eigenvalues of infinite multiplicity.

Annular Shape of Window

- Modes method

$$\psi_{I}^{k,m}(r,\theta,z)=\sum_{n=1}^{\infty}c_nR_{I}^{m,n}(r)\,e^{im\theta}\sin\left(\frac{\pi n}{d}z\right),$$

$$\psi_{II}^{k,m}(r,\theta,z)=\sum_{n=1}^{\infty}\left(d_nR_{IIa}^{m,n}(r)+e_nR_{IIb}^{m,n}(r)\right)e^{im\theta}\cos\left(\frac{\pi(2n-1)}{2d}z\right),$$

$$\psi_{II}^{k,m}(r,\theta,z)=\sum_{n=1}^{\infty}g_nR_{III}^{m,n}(r)\,e^{im\theta}\sin\left(\frac{\pi n}{d}z\right)$$

- Matching in outer radius and inner radius

$$\sum_{n=1}^{\infty}c_n\sin\left(\frac{\pi n}{d}z\right)=\sum_{n=1}^{\infty}(d_n+e_n)\cos\left(\frac{\pi(2n-1)}{2d}z\right),$$

$$d_j+e_j=\sum_{n=1}^{\infty}c_n\left(\cos\left(\frac{\pi(2j-1)}{2d}z\right),\sin\left(\frac{\pi n}{d}z\right)\right)$$

- Transfer to matrix equation $\mathbb{M}\cdot\vec{c}=0$

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