1. Schur test : $(X, \mathfrak{M}, \mu)$

$$p \in (1,\infty), \quad T: L^p \to L^p, \quad Tf(x) := \int_X K(x,y) f(y)\, d\mu(y)$$

Thm.: $K: X \times X \to [0,\infty)$ meas., $q,p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

If $\exists A,B \geq 0$ $\exists h: X \to (0,\infty)$ meas. such that

$$\int K(x,y) h^q(y)\, d\mu(y) \leq A h^q(x) \quad \mu\text{-a.e. } x \in X,$$

$$\int K(x,y) h^p(x)\, d\mu(x) \leq B h^p(y) \quad \mu\text{-a.e. } y \in X.$$

Then $T \in \mathcal{B}(L^p)$ and $\|T\| \leq A^{\frac{1}{q}} B^{\frac{1}{p}}$.

Proof: $f \in L^p$, $x \in X$

$$|Tf(x)| \leq \int \underbrace{K(x,y)}_{= K^{\frac{1}{q}} K^{\frac{1}{p}}} h(y) h^{-1}(y) |f(y)|\, d\mu(y) \overset{H}{\leq} \Big(\overbrace{\int K(x,y) h^q(y)\, d\mu(y)}^{\leq A h^q(x)} \Big)^{1/q}$$

$$\times \Big(\int K(x,y) h^{-p}(y) |f(y)|^p\, d\mu(y) \Big)^{1/p}$$

$$\int |Tf(x)|^p\, d\mu(x) \leq A^{\frac{p}{q}} \int h^p(x) \Big(\int K(x,y) h^{-p}(y) |f(y)|^p\, d\mu(y) \Big)\, d\mu(x)$$

$$\overset{T\text{-}F}{=} A^{\frac{p}{q}} \int h^{-p}(y) |f(y)|^p \Big(\underbrace{\int K(x,y) h^p(x)\, d\mu(x)}_{\leq B h^p(y)} \Big)\, d\mu(y)$$

$$\leq A^{\frac{p}{q}} \cdot B \int |f(y)|^p\, d\mu(y).$$

□

$\mu = \sum_{n=1}^{\infty} \delta_{\{n\}}$, $p=q=2$ $\rightarrow$

Cor.: $A = (a_{m,n})_{n,m=1}^{\infty}$, $a_{m,n} = a_{n,m} \geq 0$. If $\exists C \geq 0$ and $\exists h_n > 0$ s.t.
(Schur, 1911)

$$\sum_{n=1}^{\infty} a_{m,n} h_n \leq C h_m \quad \forall m \in \mathbb{N}.$$

Then $A \in \mathcal{B}(\ell^2)$ and $\|A\| \leq C$.

2. Hilbert matrix

Hilbert's ineq.: $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_n a_m}{n+m} \le C \cdot \sum_{n=1}^{\infty} a_n^2 \quad \forall a \in \ell^2(\mathbb{N})$ real

Hilbert 1901: $C \le 2\pi$ (published in 1908 in PhD diss. of H. Weyl)

(ℓ^p-generalization by Hardy & H. Riesz in 1925)

Schur 1911: $C = \pi$ optimal

$\Updownarrow$

HI: $\langle a, Ha \rangle \le \|H\| \, \|a\|^2$, $\quad H_{mn} = \frac{1}{m+n}$

Schur res. $\Leftrightarrow$ $\boxed{\|H\| = \pi.}$

3. Proof of $\|H\| \le \pi$: Schur test with $h_n := \frac{1}{\sqrt{n}}$ (& $C = \pi$)

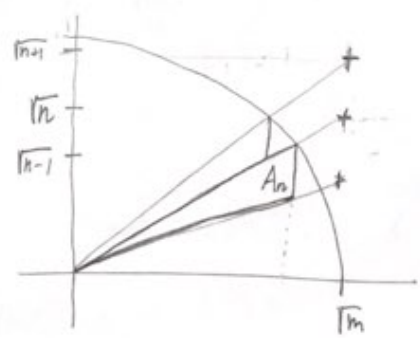
$$\sum_{n=1}^{\infty} H_{mn} h_n = \sum_{n=1}^{\infty} \frac{1}{m+n}\frac{1}{\sqrt{n}} \overset{?}{\le} \pi \frac{1}{\sqrt{m}} \Leftrightarrow \sum_{n=1}^{\infty} \frac{1}{m+n}\sqrt{\frac{m}{n}} \overset{(*)}{\le} \pi \quad \forall m \ge 1$$

Quarter-circle lemma $\equiv (*)$

area ~ (easy) $\quad A_n = \frac{1}{2}\sqrt{\frac{m^3}{m+n}}\left(\sqrt{\frac{n}{m+n}} - \sqrt{\frac{n-1}{m+n}}\right)$

$$\sum_{1}^{\infty} A_n \le \frac{\pi m}{4}$$

$$\sqrt{n} - \sqrt{n-1} \overset{MVT}{=} \frac{1}{2\sqrt{q}} \ge \frac{1}{2\sqrt{n}}$$



$$\Rightarrow \sum_{n=1}^{\infty} \frac{\sqrt{m}}{m+n}\frac{1}{\sqrt{n}} \le \pi$$

□

4. Proof of $\|H\| \ge \pi$: $\quad h_n(\varepsilon) := n^{-1/2-\varepsilon}$, $\varepsilon > 0$

We show $\lim_{\varepsilon \to 0+} \left(\cancel{\frac{\|Hh(\varepsilon)\|^2}{\|h(\varepsilon)\|^2}} = \right) \frac{\langle h(\varepsilon), Hh(\varepsilon)\rangle}{\|h(\varepsilon)\|^2} = \pi$

$$\|h(\varepsilon)\|^2 = \sum_{n=1}^{\infty} \frac{1}{n^{1+2\varepsilon}} = \int_1^{\infty} \frac{dx}{x^{1+2\varepsilon}} + \underset{\varepsilon \to 0+}{\mathcal{O}(1)} = \left[-\frac{x^{-2\varepsilon}}{2\varepsilon}\right]_1^{\infty} + \mathcal{O}(1) = \frac{1}{2\varepsilon} + \mathcal{O}(1).$$

$$\langle h(\varepsilon), Hh(\varepsilon)\rangle = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} n^{-1/2-\varepsilon}\frac{1}{n+m} m^{-1/2-\varepsilon}$$

$$= \int_1^{\infty}\int_1^{\infty} \frac{dx\,dy}{x^{1/2+\varepsilon}(x+y)y^{1/2+\varepsilon}} + \mathcal{O}(1) \quad = \left| \begin{matrix} \frac{y}{x} = u \\ dy = x\,du \end{matrix} \right|$$

$$= \int_1^{\infty} \frac{1}{x^{1+2\varepsilon}} \int_{\frac{1}{x}}^{\infty} \frac{du}{u^{1/2+\varepsilon}(1+u)} + \mathcal{O}(1) \quad = (*)$$

$$\int_0^{1/x} \frac{du}{u^{1/2+\varepsilon}(1+u)} \le \int_0^{1/x} u^{-1/2-\varepsilon}\,du = \left[\frac{u^{1/2-\varepsilon}}{1/2-\varepsilon}\right]_0^{1/x}$$

$$= \frac{1}{1/2-\varepsilon}\,\frac{1}{x^{1/2-\varepsilon}} \quad \left(\cancel{= \mathcal{O}(1),\ \varepsilon \to 0+}\right)$$

$$\int_1^{\infty} \frac{1}{1/2-\varepsilon}\,\frac{1}{x^{3/2+\varepsilon}}\,dx = \frac{1}{1/2-\varepsilon}\left[\frac{x^{-1/2+\varepsilon}}{-1/2+\varepsilon}\right]_1^{\infty} = \frac{1}{1/4-\varepsilon^2} = \mathcal{O}(1) \quad \varepsilon \to 0+$$

$$(*) = \int_1^{\infty} \frac{1}{x^{1+2\varepsilon}} \int_0^{\infty} \frac{du}{u^{1/2+\varepsilon}(1+u)}\,dx + \mathcal{O}(1)$$

$$= \left[\frac{x^{-2\varepsilon}}{-2\varepsilon}\right]_1^{\infty} \underbrace{\int_0^{\infty} \frac{du}{u^{1/2}(1+u)}}_{=\pi} + \mathcal{O}(1) = \frac{\pi}{2\varepsilon} + \mathcal{O}(1) \quad \varepsilon \to 0+$$

$$\frac{\langle h(\varepsilon), Hh(\varepsilon)\rangle}{\|h(\varepsilon)\|^2} = \frac{\pi/2\varepsilon + \mathcal{O}(1)}{1/2\varepsilon + \mathcal{O}(1)} \xrightarrow{\varepsilon \to 0+} \pi .$$

$\square$

Exer.: (to check)

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m a_n}{\max(m,n)} \le 4\sum_{n=1}^{\infty} a_n^2 \qquad \forall a \in \ell^2(\mathbb{N}) \text{ real.}$$

$$L_{m,n} = \frac{1}{\max(m,n)}, \quad \|L\| = 4.$$

$$C = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \\ \vdots & & & \ddots \end{bmatrix}$$

Cesàro matrix

$$L = CC^*, \quad \hat{L} := C^*C \Rightarrow \|\hat{L}\| = \|C\|^2 = \|L\| = 4$$

$$\hat{L}_{m,n} = \sum_{k=\max(m,n)}^{\infty} \frac{1}{k^2}$$

$$\langle a, \hat{L}a\rangle = \underbrace{\sum_{n=1}^{\infty}\left(\frac{a_1+\cdots+a_n}{n}\right)^2 \le 4\sum_{n=1}^{\infty} a_n^2}_{} \quad \underline{\text{Hardy ineq.}}$$