Michele Zaccaron

# Shape sensitivity analysis of a Maxwell's cavity problem

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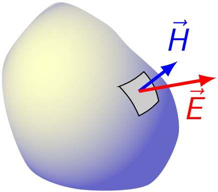
Based on joint work with P.D. Lamberti

# Introduction

The time-harmonic Maxwell's equations in a cavity $\Omega$ of $\mathbb{R}^3$ read as follows:

$$\begin{aligned} &\operatorname{curl} E = \mathrm{i}\,\omega\mu H, && \operatorname{curl} H = -\mathrm{i}\,\omega\varepsilon E \\ &\nu \times E = 0, && \nu \cdot H = 0 \end{aligned}$$




Then $\operatorname{curl} \mu^{-1} \operatorname{curl} E = \mathrm{i}\,\omega \operatorname{curl} H = -\mathrm{i}^2\omega^2\varepsilon E = \omega^2\varepsilon E$. Normalizing $\mu = \varepsilon = 1$, we end up with the following eigenvalue problem

$$(\mathcal{M}) \quad \begin{cases} \operatorname{curl}\operatorname{curl} u = \lambda u & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ \nu \times u = 0 & \text{on } \partial\Omega. \end{cases}$$

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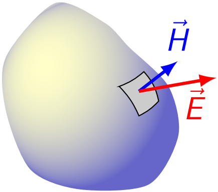


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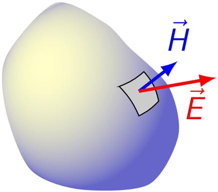


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Variational formulation: find $\lambda \geq 0$ and $u \in X$ such that

$$(\mathcal{M}) \qquad \int_\Omega \operatorname{curl} u \cdot \operatorname{curl} v \, dx = \lambda \int_\Omega u \cdot v \, dx \quad \forall v \in X$$

$$H(\operatorname{curl}, \Omega) = \{u \in L^2(\Omega)^3 : \operatorname{curl} u \in L^2(\Omega)^3\}$$

$$H_0(\operatorname{curl}, \Omega) = \{u \in L^2(\Omega)^3 : \operatorname{curl} u \in L^2(\Omega)^3, \nu \times u|_{\partial\Omega} = 0\} = \overline{\mathcal{C}_c^\infty(\Omega)^3}^{H(\operatorname{curl},\Omega)}$$

$$X_N(\Omega) = \{u \in L^2(\Omega)^3 : \operatorname{curl} u, \operatorname{div} u \in L^2(\Omega), \nu \times u|_{\partial\Omega} = 0\} \overset{\subset}{\rightarrow} L^2(\Omega)^3$$

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# A few first properties

The spectrum is discrete composed of eigenvalues of finite multiplicity

$$0 \le \lambda_1[\Omega] \le \lambda_2[\Omega] \le \cdots \le \lambda_j[\Omega] \le \cdots \nearrow +\infty$$

and we have the standard min-max characterization

$$\lambda_j[\Omega] = \min_{\substack{V \subset X_N(\Omega) \\ \dim V = j}} \max_{u \in V \setminus \{0\}} \frac{\int_\Omega |\operatorname{curl} u|^2 + |\operatorname{div} u|^2 dx}{\int_\Omega |u|^2 \, dx}.$$

The existence of the zero eigenvalues depends on topological properties of $\Omega$. Indeed

$$K = \{u \in L^2(\Omega)^3 : \operatorname{curl} u = 0, \operatorname{div} u = 0, \nu \times u|_{\partial\Omega} = 0\}$$

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# Product domains

If $\Omega = \omega \times I$ for some simply connected domain $\omega$ of $\mathbb{R}^2$ and some finite interval $I \subset \mathbb{R}$. Then the Maxwell eigenvalues span the set

$$\{\lambda_m^\omega + \mu_n^I\}_{m\geq 1, n\geq 0} \cup \{\mu_n^\omega + \lambda_m^I\}_{m\geq 1, n\geq 1}$$

where

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...or monotonicity:

$$A \subseteq B \implies d_j^{-\Delta,\mathfrak{D}}(A) \geq d_j^{-\Delta,\mathfrak{D}}(B)$$

The monotonicity principle does not hold for Neumann Laplacian, and neither for Maxwell. On a parallelepiped the first Maxwell eigenvalue coincide with the first (positive) Dirichlet Laplacian eigenvalue in $\mathbb{R}^2$ of the largest face. That is, if $\Omega = (0,\ell_1) \times (0,\ell_2) \times (0,\ell_3)$ with $\ell_1 \geq \ell_2 \geq \ell_3$ then

$$m_1^\Omega = \frac{\pi^2}{\ell_1^2} + \frac{\pi^2}{\ell_2^2}$$

If we consider the halved parallelepiped $\widehat{\Omega} = (0,\ell_1/2) \times (0,\ell_2/2) \times (0,\ell_3/2) \subset \Omega$ it is immediate to see that

$$\hat{m}_1 = 4m_1^\Omega > m_1^\Omega$$

Since $\tilde{\ell}_1 > \ell_1$ and $\tilde{\ell}_2 = \ell_2$, then

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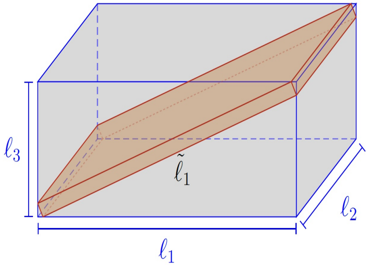


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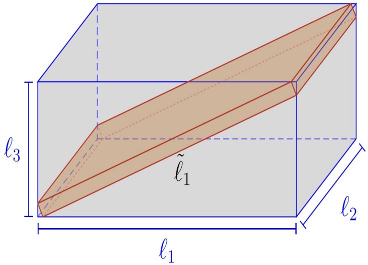


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# Short history

**Question**

What can we say about the behaviour of the eigenvalues $\lambda_j[\Omega]$ w.r.t. perturbations of the domain $\Omega$? In particular, can we provide a formula for the shape derivative?

- Hadamard variation: in the beginning of last century the work of Hadamard[1] on shape variations for the Dirichlet Laplacian.
- The same Maxwell problem is considered in Jimbo [2]: uni-parametric perturbations, simple eigenvalues.
- Our shape derivative formula coincides with the one found in "Electromechanics" (*Denki Rikigaku* - Hirakawa '73)

It is of "different type" from Jimbo's.

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としてもよい。そこで

$$\frac{\omega^2-\omega_m{}^2}{\omega_m{}^2} = -\frac{\iint(\varepsilon|\mathbf{E}_m|^2-\mu|\mathbf{H}_m|^2)\,\delta n\,dS}{\varepsilon\iiint_V|\mathbf{E}_m|^2 dV} \qquad (4\text{-}88)$$

が得られる。ただし(4-87)の右辺の評価において S と S′ の間が十分に近

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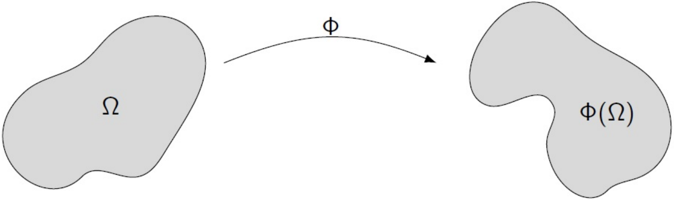
Fix a domain $\Omega \subset \mathbb{R}^3$ and consider a class of diffeomorpshims $\Phi$ on $\Omega$.




We consider the eigenvalue problem on $\Phi(\Omega)$. Its spectrum is

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The general idea is to get information about minimization/maximization of eigenvalues, under some physically or mathematically reasonable constraints. For example, we are interested in extremum problems of this type

$$\min_{\mathrm{Vol}[\Phi(\Omega)]=const.} \lambda_j[\Phi] \quad \text{or} \quad \max_{\mathrm{Vol}[\Phi(\Omega)]=const.} \lambda_j[\Phi]$$

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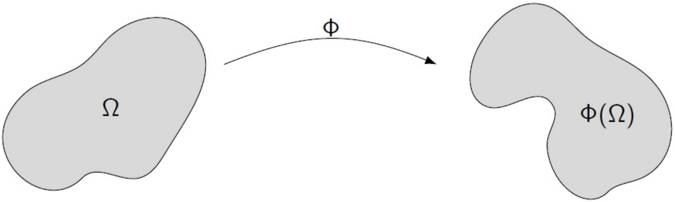


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# Shape perturbation

**Case:** the eigenvalue is simple, and we are in a particular one-parametric case where the variation acts on the boundary of $\Omega$ as follows ($\rho \in C^1(\partial\Omega)$)

$$\partial\Omega_\epsilon = \{\xi + \epsilon\rho(\xi)\nu(\xi) \in \mathbb{R}^3 : \xi \in \partial\Omega\}.$$

**Theorem (Lamberti, Z.)**

**i)** *The map $\Phi \mapsto \lambda_j[\Phi]$ is real-analytic;*

**ii)** *Hadamard formula:*

$$\left.\frac{d\lambda_j(\epsilon)}{d\epsilon}\right|_{\epsilon=0} = \int_{\partial\Omega} \left(\lambda_k(0)\left|u^{(j)}\right|^2 - \left|\operatorname{curl} u^{(j)}\right|^2\right)\rho\, d\sigma$$

*where $u^{(j)}$ is the eigenvector associated to $\lambda_j(0)$ normalized in $L^2(\Omega)^3$.*

Shape sensitivity analysis for electromagnetic cavities

[3][A real analyticity result for symmetric functions of the eigenvalues of a domain dependent Dirichlet problem for the Laplace operator]

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Shape sensitivity analysis for electromagnetic cavities

[3] [A real analyticity result for symmetric functions of the eigenvalues of a domain dependent Dirichlet problem for the Laplace operator]

# Shape perturbation

**Case:** the eigenvalue is simple, and we are in a particular one-parametric case where the variation acts on the boundary of $\Omega$ as follows ($\rho \in C^1(\partial\Omega)$)

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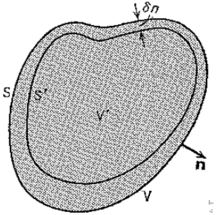
# Comparing formulas

- Hirakawa 1973 ( magnetic field $H = -\mathrm{i}\mu^{-1}\varepsilon \,\mathrm{curl}\, E/\sqrt{\lambda}$)

$$\frac{\lambda - \lambda(0)}{\lambda(0)} = \frac{\int\int(\varepsilon|E|^2 - \mu|H|^2)\delta n\, dS}{\varepsilon\int\int\int |E|^2 dV}$$




- Jimbo 2013 ($K(x)$ is the Gaussian curvature at $x \in \partial\Omega$)

$$\left.\frac{d\lambda(\epsilon)}{d\epsilon}\right|_{\epsilon=0} = \int_{\partial\Omega}\left(|DE|^2 - 2\left|\frac{\partial E}{\partial\nu}\right|^2 + 2\left(K(x) - \lambda(0)\right)|E|^2\right)\rho\, d\sigma$$
$$+ 2\int_{\partial\Omega}(E\cdot\nu)(\mathrm{curl}\, E \times \nabla_\Gamma \rho)\cdot\nu\, d\sigma$$

- Lamberti, Z. 2020

$$\left.\frac{d\lambda(\epsilon)}{d\epsilon}\right|_{\epsilon=0} = \int_{\partial\Omega}\left(\lambda(0)\;\varepsilon E\cdot E - \mu^{-1}\,\mathrm{curl}\, E\cdot\mathrm{curl}\, E\right)\rho\, d\sigma$$

# Corollaries

**i)** Rellich-Pohozaev identity ($\lambda$ can be multiple):

$$\lambda = \frac{1}{2}\int_{\partial\Omega} \left( |\operatorname{curl} u|^2 - \lambda |u|^2 \right)(x \cdot \nu)\, d\sigma$$

**ii)** Characterization of critical shapes for the (elementary symmetric functions of the) eigenvalues w.r.t. isovolumetric and isoperimetric perturbations. Let $\Omega$ a $C^2$ bounded domain of $\mathbb{R}^3$ such that $\lambda_j[\Omega]$ is simple, and denote with $u^{(j)}$ its associated (normalized) eigenfield. Then

| | | |
|---|---|---|
| fixed volume | $\lambda_j[\Omega]\, |u^{(j)}|^2 - |\operatorname{curl} u^{(j)}|^2 = const$ | on $\partial\Omega$ |
| fixed perimeter | $\lambda_j[\Omega]\, |u^{(j)}|^2 - |\operatorname{curl} u^{(j)}|^2 = const \cdot \mathcal{H}$ | on $\partial\Omega$ |

where $\mathcal{H}$ is the mean curvature. Balls are critical shapes for both isovolumetric and isoperimetric constraints.
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# Some open problems

- What is the correct type of constraints for Maxwell?
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Thanks for your attention!

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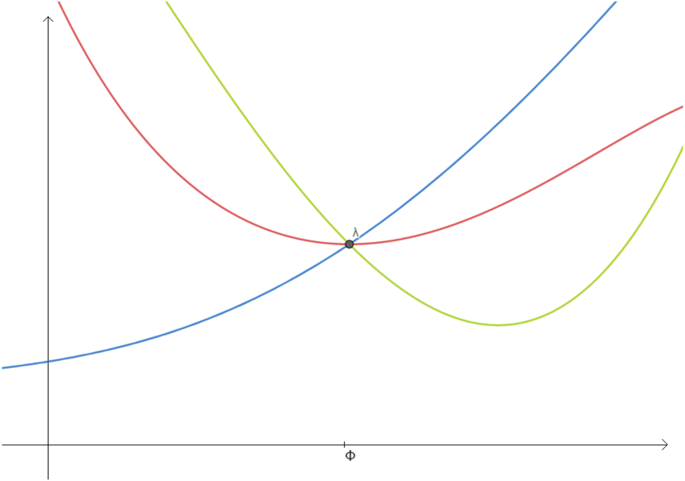
# Shape perturbation

**Problem**

If we have a multiple eigenvalue, a perturbation of the domain may split its multiplicity, causing angular bifurcation phenomena. The best we can obtain is Lipschitz continuity.



# Bifurcations

This problem can be overcome when dealing with uni-parametric families of perturbations $\{\Omega_\epsilon\}_{\epsilon>0}$ of $\Omega$. But even when we have only two parameters

Example:

$$A(t,r) = \begin{pmatrix} t & r \\ r & -t \end{pmatrix}$$

$$\lambda_1[t,r] = \sqrt{t^2 + r^2}$$

$$\lambda_2[t,r] = -\sqrt{t^2 + r^2}$$

At the point $(t,r) = (0,0)$ the eigenvalues are NOT differentiable.

The symmetric functions of the eigenvalues are differentiable.

$$\lambda_1[t,r] + \lambda_2[t,r] = 0$$

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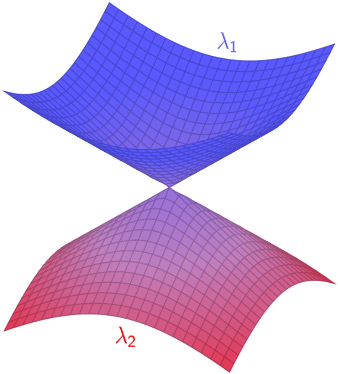


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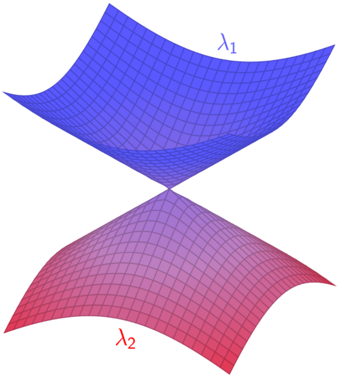


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# Shape perturbation

**Idea:** In the same spirit of Lamberti&Lanza[3] '04, we consider the elementary symmetric functions of the eigenvalues. Let $F$ be a finite subset of $\mathbb{N}$ and let $s \in \{1, \ldots, |F|\}$. Then

$$\Lambda_{F,s}[\Phi] := \sum_{\substack{j_1,\ldots,j_s \in F \\ j_1 < \cdots < j_s}} \lambda_{j_1}[\Phi] \cdots \lambda_{j_s}[\Phi]$$

**Theorem (Lamberti, Z.)**

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# Gaffney-Friedrichs inequality

Recall $X_N(\Omega) = \{u \in L^2(\Omega) : \operatorname{curl} u, \operatorname{div} u \in L^2(\Omega), \nu \times u = 0 \text{ on } \partial\Omega\}$
Gaffney inclusion: $X_N(\Omega) \subset H^1(\Omega)^3$.
Gaffney inequality: for all $u \in X_N(\Omega)$

$$||Du||^2_{L^2(\Omega)^{3\times 3}} \leq C \left( ||\operatorname{div} u||^2_{L^2(\Omega)} + ||\operatorname{curl} u||^2_{L^2(\Omega)^3} + ||u||^2_{L^2(\Omega)^3} \right)$$

Dirichlet Laplacian: $\begin{cases} -\Delta\varphi = f, & \text{in } \Omega, \\ \varphi = 0, & \text{on } \partial\Omega, \end{cases}$

If $\Omega$ is at least Lipschitz

Gaffney inclusion $\Longleftrightarrow$ $H^2$-regularity for the Dirichlet Laplacian

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M. Sh. Birman and M. Z. Solomyak. The Maxwell operator in domains with a nonsmooth boundary. '87

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# Weyl law

In fact, it was Weyl[6] himself the first to obtain it for Maxwell

$$N_{\mathcal{M}}(\lambda) \sim \frac{|\Omega|}{3\pi^2}\lambda^{3/2}$$

Dirichlet/Neumann Laplacian in $\mathbb{R}^3$

$$N_{\mathcal{L}_{\mathcal{D},\mathcal{N}}}(\lambda) \sim \frac{|\Omega|}{2\cdot 3\pi^2}\lambda^{3/2}$$

Pólya conjectured in 1961 that the Weyl estimate of large eigenvalues should be a strict lower bound for each of the Dirichlet eigenvalues of a domain, and an upper bound for the Neumann eigenvalues.
Berezin, Li-Yau proved an averaged version of the conjecture for Dirichlet:

$$\frac{1}{k}\sum_{j=1}^{k} d_j \geq \frac{3}{5}(2\cdot 3\pi^2)2/3\frac{k}{|\Omega|}^{2/3},$$

Kröger for Neumann

$$\frac{1}{k}\sum_{j=1}^{k} \mu_j \leq \frac{3}{5}(2\cdot 3\pi^2)2/3\frac{k}{|\Omega|}^{2/3}.$$

[6] [Über das Spectrum der Hohlraumstrahlung (1912)]