Superbrief introduction to SUPERSYMMETRIC METHODS

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The Dirac Equation

(Bernd Thaller, Springer-Verlag, 1992)

Abstract setting

$\mathcal{H}$ … Hilbert space

τ … unitary involution ($\Rightarrow$ s.a.), $\tau^*\tau = \tau^*\tau = \tau^2 = I$

$\Rightarrow \sigma(\tau) = \sigma_p(\tau) \subset \{-1, 1\}$

$\rightarrow$ assume $\sigma_p(\tau) = \{-1, 1\}$ with eigenspaces $\mathcal{H}_{-1}, \mathcal{H}_{+}$

$\Rightarrow \boxed{\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-}$

$P_\pm := \frac{1}{2}(I \pm \tau)$ … OG projections on $\mathcal{H}_\pm$

Ex.: $\mathcal{H} = L^2(\mathbb{R}; \mathbb{C}^2)$, $\tau = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \mathcal{H} : \psi_i \in L^2(\mathbb{R}; \mathbb{C}) \rightarrow \tau \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ -\psi_2 \end{pmatrix}$

$\mathcal{H}_+ = \left\{ \begin{pmatrix} \psi \\ 0 \end{pmatrix} : \psi \in L^2(\mathbb{R}) \right\}$, $\mathcal{H}_- = \left\{ \begin{pmatrix} 0 \\ \psi \end{pmatrix} : \psi \in L^2(\mathbb{R}) \right\}$

$P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\qquad P_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Abstract Dirac Operator

$H = H^*$ on $\mathcal{D}(H) \subset \mathcal{H}$: $\tau \mathcal{D}(H) = \mathcal{D}(H)$

H can be uniquely written as $H = H_{odd} + H_{even}$ s.t.
$H_{odd/even}$ are symm. on $\mathcal{D}(H)$ and
$\{H_{odd}, \tau\} = 0$, $[H_{even}, \tau] = 0$ on $\mathcal{D}(H)$.

Proof: $P_\pm \mathcal{D}(H) \subset \mathcal{D}(H)$, $P_+ + P_- = I$

$$H = \underbrace{(P_+ H P_+ + P_- H P_-)}_{H_{even}} + \underbrace{(P_+ H P_- + P_- H P_+)}_{H_{odd}}$$

symmetry ✓

$$\{H_{odd}, \tau\} = P_+ H \underbrace{P_- \tau}_{\frac{1}{2}(I-\tau)\tau} + P_- H \underbrace{P_+ \tau}_{P_+} + \underbrace{\tau P_+}_{P_+} H P_- + \underbrace{\tau P_-}_{-P_-} H P_+ = 0$$

$$\frac{1}{2}(I-\tau)\tau = \frac{1}{2}(\tau - I) = -P_-$$

uniqueness : $H = \tilde{H}_{odd} + \tilde{H}_{even}$: $\{\tilde{H}_{odd}, \tau\} = 0$, $[\tilde{H}_{even}, \tau] = 0$

$$0 = H_{odd} - \tilde{H}_{odd} + (H_{even} - \tilde{H}_{even}) \; \ldots \text{ commutes with } \tau$$

$$[H_{even} - \tilde{H}_{even}, \tau] = 0 \implies [\underbrace{H_{odd} - \tilde{H}_{odd}}_{\overset{!!}{=}\, T}, \tau] = 0$$

$$\tau \,/\, \tau T = T\tau \overset{\{T, \tau\} = 0}{=} -\tau T$$

$$T = -T \implies T = 0 \implies H_{odd} = \tilde{H}_{odd} \implies H_{even} = \tilde{H}_{even}$$

$\square$

H_{odd} ... odd/fermionic part   e e u d

H_{even} ... even/bosonic part γ γ

if $\boxed{Q = Q_{odd}}$ then Q is called a supercharge (w.r.t. τ) :

$$Q = Q^*, \quad \tau \mathcal{D}(Q) = \mathcal{D}(Q), \quad \{\tau, Q\} = 0 \text{ on } \mathcal{D}(Q)$$

Standard representation

$\psi \in \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$

$\psi_+ + \psi_- = \psi \equiv \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$

$\Rightarrow \tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (= \sigma_3)$

$H = P_+ H P_+ + \ldots = \begin{pmatrix} H_+ & H_{+-} \\ H_{-+} & H_- \end{pmatrix}$

$H_{even} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}, \quad H_{odd} = \begin{pmatrix} 0 & H_{+-} \\ H_{-+} & 0 \end{pmatrix}$

is. s.a./e.s.a./closed iff H_+ & H_- are both s.a./e.s.a./closed

?

Lemma: $D_\pm : \mathcal{D}(D_\pm) \subset \mathcal{H}_\pm \to \mathcal{H}_\mp$ ... densely defined closable,

$Q := \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}$ on $\mathcal{D}(D_+) \oplus \mathcal{D}(D_-)$.

Then $Q^* = \begin{pmatrix} 0 & D_+^* \\ D_-^* & 0 \end{pmatrix}, \quad Q^{**} = \begin{pmatrix} 0 & D_-^{**} \\ D_+^{**} & 0 \end{pmatrix}$.

Proof: $J: \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \to \mathcal{K} := \mathcal{H}_- \oplus \mathcal{H}_+$

$J\begin{pmatrix} f_+ \\ f_- \end{pmatrix} := \begin{pmatrix} f_- \\ f_+ \end{pmatrix}, \quad f_\pm \in \mathcal{H}_\pm$ $\Big] \to$ unitary isomorphism

$$Q = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} D_- & 0 \\ 0 & D_+ \end{pmatrix}}_{\overset{!!}{P}} \overset{\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}}{\overset{\shortparallel}{J}}$$

P ... closed/closable iff D_- & D_+ are closed/closable

$\to$ this property carries over to Q

$$Q^* = (PJ)^* = J^* P^* = J^{-1}\begin{pmatrix} D_-^* & 0 \\ 0 & D_+^* \end{pmatrix} = \begin{pmatrix} 0 & D_+^* \\ D_-^* & 0 \end{pmatrix}$$

$$Q^{**} = \ldots = \begin{pmatrix} 0 & D_-^{**} \\ D_+^{**} & 0 \end{pmatrix}$$

$\blacksquare$

Corollary: $Q = Q^* \iff D_- = D_+^*$ & $D_+ = D_-^* \; (= D_+^{**})$

$\iff D_- = D_+^*$ & $D_+ = \overline{D_+}$

$\iff Q = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}$ on $\mathcal{D}(D) \oplus \mathcal{D}(D^*)$

with $D = \overline{D}$

Theorem [von Neumann]: D ... densely defined closed.
Then D^*D is densely defined and self-adjoint.

Remark: For unbounded op. A, B we only have $(AB)^* \supset B^*A^*$

Proof [unpublished remark of Edward Nelson]:

$$Q = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix} = Q^*$$

$$Q^2 = \begin{pmatrix} D^*D & 0 \\ 0 & DD^* \end{pmatrix}$$

$$\mathcal{D}(Q^2) = \{ \psi \in \mathcal{D}(Q) : Q\psi \in \mathcal{D}(Q) \} = \mathcal{D}(D^*D) \oplus \mathcal{D}(DD^*)$$

$\Rightarrow$ by the spectral theorem / functional calculus, Q^2 is densely defined and s.a.

$\Rightarrow$ the same holds true for D^*D $\blacksquare$

Polar decomposition

$$Q = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix} = Q^* \quad \longrightarrow \quad Q = \underbrace{|Q|}_{\overset{\shortparallel}{(Q^2)^{1/2}}} \underbrace{\operatorname{sgn} Q}_{\overset{\shortparallel}{Q|Q|^{-1} = |Q|^{-1}Q \text{ on } (\ker Q)^\perp}} = \operatorname{sgn} Q \, |Q|$$

$\leftrightarrow$ unitary

(extended by zero on $\ker Q$ to a partial isometry)

$$(Q^2)^{1/2} = \begin{pmatrix} (D^*D)^{1/2} & 0 \\ 0 & (DD^*)^{1/2} \end{pmatrix}$$

$$Q = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix} \begin{pmatrix} (D^*D)^{1/2} & 0 \\ 0 & (DD^*)^{1/2} \end{pmatrix} = \begin{pmatrix} 0 & S^*(DD^*)^{1/2} \\ S(D^*D)^{1/2} & 0 \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}$$

$S = D((D^*D)^{1/2})^{-1}$ on $(\ker D)^\perp$ (extended by 0)

$$\ker Q = \begin{pmatrix} \ker D \\ \ker D^* \end{pmatrix}$$

$\leftrightarrow$ S ... unitary : $(\ker D)^\perp$ onto $(\ker D^*)^\perp$

$$\begin{pmatrix} D^*D & 0 \\ 0 & DD^* \end{pmatrix} = Q^2 = \operatorname{sgn} Q \; Q^2 \; \operatorname{sgn} Q = \begin{pmatrix} S^*DD^*S & 0 \\ 0 & SD^*DS^* \end{pmatrix}$$

on $(\ker Q)^\perp$

Spectral supersymmetry

$\Rightarrow$ D^*D on $(\ker D)^\perp$ is unitarily equiv. to DD^* on $(\ker D^*)^\perp$

$\ker D = \ker D^*D$ $\qquad\qquad$ $\ker D^* = \ker DD^*$

Lemma: A on $(\ker A)^\perp$ is unitarily equiv. to B on $(\ker B)^\perp$.

Then $\sigma(A)\setminus\{0\} = \sigma(B)\setminus\{0\}$.

Proof: $\Leftrightarrow$ $\varrho(A)\cup\{0\} = \varrho(B)\cup\{0\}$

$\rightarrow$ take $\lambda \in \varrho(A)\setminus\{0\}$: $(\forall g\in\mathcal{H})\,(\exists_1 \psi\in\mathcal{D}(A))\,((A-\lambda)\psi = g)$

$$(B-\lambda)\varphi = f \in \mathcal{H}$$
$$(SAS^*-\lambda)\varphi = f \qquad (S \ldots \text{unitary} : (\ker A)^\perp \to (\ker B)^\perp)$$
$$(A-\lambda)S^*\varphi = S^*f$$

$\rightarrow$ $\exists_1 \psi \in \mathcal{D}(A) : (A-\lambda)\psi = S^*f$

$\varphi = S\psi \ldots \exists_1$ for $\psi\in(\ker A)^\perp$

if $\psi \in \ker A\setminus\{0\}$ then $(A-\lambda)\psi = \underset{\neq 0}{-\lambda\psi} = S^*f \in (\ker A)^\perp$

$\Rightarrow$ no way $\blacksquare$

$$\boxed{\sigma(DD^*)\setminus\{0\} = \sigma(D^*D)\setminus\{0\}}$$

Spectral supersymmetry - eigenvalues

$0 \neq \lambda \in \sigma_p(D^*D)$: $D^*Df = \lambda f$

$DD^*(Df) = D(D^*Df) = \lambda Df$

and $Df \neq 0$: otherwise $D^*Df = 0 = 0f$, but $\lambda \neq 0$

$\Rightarrow \lambda \in \sigma_p(DD^*)$ and Df is an eigenvector

... and conversly, ...

$\Rightarrow$ all eigenvectors g of DD^* may be written as $g = Df$ with f an eigenvector of D^*D belonging to the same e.v. $(\neq 0)$

Dirac operators with supersymmetry

$$H = Q + M\tau$$

supercharge w.r.t. τ $\quad$ ($\{Q,\tau\}=0$)

positive s.a. : $[M,Q]=[M,\tau]=0$

$+\ M, M^{-1} \in \mathcal{B}(\mathcal{H})$ & $M\mathcal{D}(Q)=\mathcal{D}(Q)$

Ex.

$$H_{1D} = -ic\,\sigma_1 \frac{d}{dx} + \underset{0}{\underset{\vee}{m}}c^2\sigma_3 = \begin{pmatrix} mc^2 & -icd_x \\ -icd_x & -mc^2 \end{pmatrix} \quad ; \quad \tau = \sigma_3\,,\ M = mc^2 I$$

$$H_{3D} = \begin{pmatrix} mc^2 I & -ic\,\sigma\cdot\nabla \\ -ic\,\sigma\cdot\nabla & -mc^2 I \end{pmatrix} \quad ; \quad \sigma\cdot\nabla := \sum_{i=1}^{3} \sigma_i \frac{\partial}{\partial x^i}$$

Foldy-Wouthuysen trafo

$$U_{FW} := a_+ + \tau(\operatorname{sgn} Q)\,a_- \quad , \quad a_\pm := \frac{1}{\sqrt{2}}\sqrt{1 \pm M|H|^{-1}}$$

unitary

$$U_{FW} H U_{FW}^* = \tau|H| = \tau\sqrt{Q^2+M^2} =: H_{FW}$$

$$[H_{FW},\tau]=0$$

$$H_{FW}^2 = H^2$$

$$M = \begin{pmatrix} M_+ & 0 \\ 0 & M_- \end{pmatrix}$$

$$\Rightarrow \quad U_{FW} \begin{pmatrix} M_+ & D^* \\ D & -M_- \end{pmatrix} U_{FW}^* = \begin{pmatrix} \sqrt{D^*D + M_+^2} & 0 \\ 0 & -\sqrt{DD^* + M_-^2} \end{pmatrix}$$

determine $\sigma(H)$

$M = \overset{0}{m} I$: due to the spectral supersymmetry, $\sigma(D^*D) \setminus \{0\} = \sigma(DD^*) \setminus \{0\}$

+ spectral mapping theorem

$$\Rightarrow \quad \sigma\left(\sqrt{D^*D + m^2}\right) \setminus \{m\} = \sigma\left(\sqrt{DD^* + m^2}\right) \setminus \{m\}$$

$\Rightarrow$ $\boxed{\sigma(H) \text{ is symmetric w.r.t. } 0}$ (except possibly at $\pm m$)

since $D^*D \geq 0$, $\sigma(H) \cap (-m, m) = \emptyset$

Ex. $H_{1D} = -ic\sigma_1 \frac{d}{dx} + mc^2 \sigma_3$, $\mathcal{D}(H_{1D}) = H^1(\mathbb{R}; \mathbb{C}^2)$

$\Rightarrow D^*D = \left(-ic\frac{d}{dx}\right)^2 = -c^2 \frac{d^2}{dx^2}$ on $H^2(\mathbb{R}; \mathbb{C})$

$\Rightarrow \sigma(D^*D) \;(= \sigma_{ac}(D^*D)) = [0, +\infty)$

$\Rightarrow \sigma(H_{1D}) = (-\infty, -m] \cup [m, +\infty)$

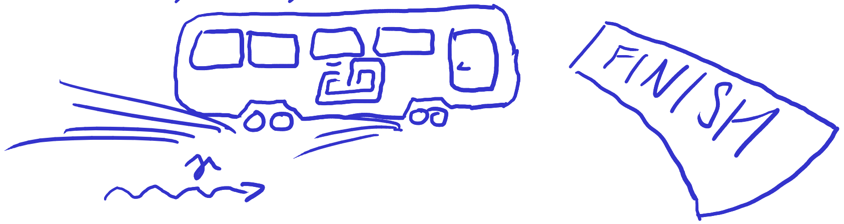
Non-relativistic limit

introducing c : $H(c) = cQ + \tau M c^2 = \begin{pmatrix} M_+ c^2 & cD^* \\ cD & -M_- c^2 \end{pmatrix}$

relativistic VS non-relativistic physics

$c > 0$ ... principal bound for the propagation speed of signals

no principal bound



$$\xrightarrow{\text{non-rel}}$$

c should be infinitely large compared to other velocities

$$\Downarrow$$

$\boxed{c \to \infty}$:

Mc^2 diverges and has no interpretation in non-rel lim.

$\Rightarrow$ subtract it!

put $H(\infty) = \frac{1}{2} M^{-1} Q^2 \equiv \frac{1}{2M} Q^2$, recall that $P_+ = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$, $P_- = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$

Theorem: For every $z \in \mathbb{C} \setminus \mathbb{R}$, there exists c_z s.t. $\forall c > c_z$

Thaller: $\pm$

$$(H_0(c) \mp Mc^2 - z)^{-1} = \left(P_\pm \pm \frac{1}{c^2} \frac{cQ + z}{2M}\right)\left(I \mp \frac{1}{c^2} \frac{z^2}{2M} (\pm H(\infty) - z)^{-1}\right)^{-1} (\pm H(\infty) - z)^{-1}.$$

Proof: $A_\pm := H_0(c) \pm Mc^2 \pm z = cQ \pm 2Mc^2 P_\pm \pm z$

$\rightarrow$ using (anti) commutation relations:

$M^{-1} \in \mathcal{B}(\mathcal{X})$

$$A_+ A_- = A_- A_+ = c^2 Q^2 - 2Mc^2 z - z^2 = \underbrace{\Big(\underbrace{\frac{Q^2}{2M} - z}_{\text{invertible}} - \underbrace{\frac{z^2}{2Mc^2}}_{\|-\|-\| \xrightarrow{c \to \infty} 0}\Big)}_{\text{invertible for all } c \text{ large enough}} 2Mc^2$$

$$A_\pm^{-1} = \frac{A_\mp}{2Mc^2} \Big(\underbrace{\frac{Q^2}{2M} - z}_{\substack{A \\ \| \\ H(\infty) - z}} - \underbrace{\frac{z^2}{2Mc^2}}_{B}\Big)^{-1} \quad : \quad (A+B)^{-1} = (I + A^{-1}B)^{-1} A^{-1}$$

$\blacksquare$

$c \to \infty$: $\left\| \frac{1}{c^2} \frac{z^2}{2M} (\pm H(\infty) - z)^{-1} \right\| \to 0 \Rightarrow \left(I \mp \frac{1}{c^2} \frac{z^2}{2M} (\pm H(\infty) - z)^{-1}\right)^{-1} \to I$

$$\frac{1}{c^2} \frac{cQ + z}{2M} = \underbrace{\frac{z}{c^2 2M}}_{\| \cdot \| \to 0} + \underbrace{\frac{1}{2M}}_{\text{bounded}} \frac{Q}{c} \quad ? \left(\frac{Q}{c}\Big(\frac{Q^2}{2M} - z - \frac{z^2}{2Mc^2}\Big)^{-1}\right)$$

bounded uniformly in c

$$B_c := Q\left(\frac{Q^2}{2M} - z - \frac{z^2}{2Mc^2}\right)^{-1} = Q\left(\frac{Q^2}{2M} - z\right)^{-1} + Q\left(\frac{Q^2}{2M} - z - \frac{z^2}{2Mc^2}\right)^{-1}\frac{z^2}{2Mc^2}\left(\frac{Q^2}{2M} - z\right)^{-1}$$

$$B_c\Big(I - \underbrace{\frac{z^2}{2Mc^2}\left(\frac{Q^2}{2M} - z\right)^{-1}}_{\|-\|-\| \xrightarrow{c\to\infty} 0}\Big) = Q\left(\frac{Q^2}{2M} - z\right)^{-1} \in \mathcal{B}(\mathcal{H})$$ by the closed graph thm.

$$B_c = Q\left(\frac{Q^2}{2M} - z\right)^{-1}\Big(I - \underbrace{\frac{z^2}{2Mc^2}\left(\frac{Q^2}{2M} - z\right)^{-1}}_{\|-\|-\| \le \frac{1}{2} \text{ for } c > c_z}\Big)^{-1}$$

$$\Rightarrow \|B_c\| \le \left\|Q\left(\frac{Q^2}{2M} - z\right)^{-1}\right\| \frac{1}{1-\frac{1}{2}} \quad \checkmark$$

$$\Downarrow$$

$$\boxed{\lim_{c\to\infty}\left(H(c) \mp Mc^2 - z\right)^{-1} = P_\pm\left(\pm H(\infty) - z\right)^{-1}}$$

Ex. $$\lim_{c\to\infty}\Big(\underbrace{-ic\,\sigma_1\frac{d}{dx} + mc^2\sigma_3}_{H_{1D}} - mc^2 - z\Big)^{-1} = \begin{pmatrix} \left(-\frac{1}{2m}\frac{d^2}{dx^2} - z\right)^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

Rem: $(H(c) \mp Mc^2 - z)^{-1}$ is analytic in $\frac{1}{c}$ at $c = +\infty$