MAFIA - the seminar you can't refuse

Hardy's inequality for the fractional powers of a discrete Laplacian

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Abstract:
We present a discrete version of Hardy's inequality for fractional powers of a discrete Laplacian. More precisely, consider the discrete Laplacian defined by $\Delta_{\mathrm{d}} f(n) = f(n+1) - 2f(n) + f(n-1)$, $n \in \mathbb{Z}$, for any sequence $f : \mathbb{Z} \to \mathbb{N}$. Let $0 < \sigma < 1/2$. Then for any f with compact support, we have

$$\sum_{n\in\mathbb{Z}} f(n)(-\Delta_{\mathrm{d}})^{\sigma} f(n) \geq 4^{\sigma} \frac{\Gamma\big(\frac{1+2\sigma}{4}\big)^2}{\Gamma\big(\frac{1-2\sigma}{4}\big)^2} \sum_{n\in\mathbb{Z}} f(n)^2 w_{\sigma}(n),$$

where $w_{\sigma} = \frac{\Gamma(|n|+\frac{1-2\sigma}{4})}{\Gamma(|n|+\frac{1+2\sigma}{4})} \frac{\Gamma(|n|+\frac{3-2\sigma}{4})}{\Gamma(|n|+\frac{3+2\sigma}{4})}$. The method of proof follows the ground state representation approach used by R.L. Frank, E.H. Lieb, and R. Seiringer in the Euclidean setting. Optimality of the Hardy weight and hence affirmative answer to the question of sharpness of the constant has been recently confirmed by M. Keller and M. Nietschmann. Higher dimensional analogues are indicated.

This is joint work with Ó. Ciaurri.