MAFIA - the seminar you can't refuse

On existence of minimizers for weighted L^p-Hardy inequalities on $C^{1,\gamma}$-domains with compact boundary

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Abstract:
Let $p \in (1,\infty)$, $\alpha \in \mathbb{R}$, and $\Omega \subsetneq \mathbb{R}^N$ be a $C^{1,\gamma}$-domain with a compact boundary $\partial\Omega$, where $\gamma \in (0,1]$. Denote by $\delta_\Omega(x)$ the distance of a point $x \in \Omega$ to $\partial\Omega$. Let $\widetilde{W}_0^{1,p;\alpha}(\Omega)$ be the closure of $C_c^\infty(\Omega)$ in $\widetilde{W}^{1,p;\alpha}(\Omega)$, where

$$\widetilde{W}^{1,p;\alpha}(\Omega) := \left\{ \varphi \in W^{1,p}_{\mathrm{loc}}(\Omega) \mid \left(\| \, |\nabla \varphi| \, \|^p_{L^p(\Omega;\delta_\Omega^{-\alpha})} + \|\varphi\|^p_{L^p(\Omega;\delta_\Omega^{-(\alpha+p)})} \right) < \infty \right\}.$$

We study the following two variational constants: the *weighted Hardy constant*

$$H_{\alpha,p}(\Omega) := \inf \left\{ \int_\Omega |\nabla\varphi|^p \delta_\Omega^{-\alpha} \mathrm{d}x \;\middle|\; \int_\Omega |\varphi|^p \delta_\Omega^{-(\alpha+p)} \mathrm{d}x = 1, \varphi \in \widetilde{W}_0^{1,p;\alpha}(\Omega) \right\},$$

and the *weighted Hardy constant at infinity*

$$\lambda^\infty_{\alpha,p}(\Omega) := \sup_{K \Subset \Omega} \inf_{W_c^{1,p}(\Omega\setminus K)} \left\{ \int_{\Omega\setminus K} |\nabla\varphi|^p \delta_\Omega^{-\alpha} \mathrm{d}x \;\middle|\; \int_{\Omega\setminus K} |\varphi|^p \delta_\Omega^{-(\alpha+p)} \mathrm{d}x = 1 \right\}.$$

We show that $H_{\alpha,p}(\Omega)$ is attained if and only if the spectral gap $\Gamma_{\alpha,p}(\Omega) := \lambda^\infty_{\alpha,p}(\Omega) - H_{\alpha,p}(\Omega)$ is strictly positive. Moreover, we obtain tight decay estimates for the corresponding minimizers. Furthermore, when Ω is *bounded* and $\alpha + p = 1$, then $\lambda^\infty_{1-p,p}(\Omega) = 0$ (no spectral gap) and the associated operator $-\Delta_{1-p,p}$ is null-critical in Ω with respect to the weight δ_Ω^{-1}, whereas, if $\alpha + p < 1$, then $\lambda^\infty_{\alpha,p}(\Omega) = \left|\frac{\alpha+p-1}{p}\right|^p > 0 = H_{\alpha,p}(\Omega)$ (positive spectral gap) and $-\Delta_{\alpha,p}$ is positive-critical in Ω with respect to the weight $\delta_\Omega^{-(\alpha+p)}$.

This is a joint work with Ujjal Das and Baptiste Devyver.